Chapter 2

Macromechanics of a Composite Lamina

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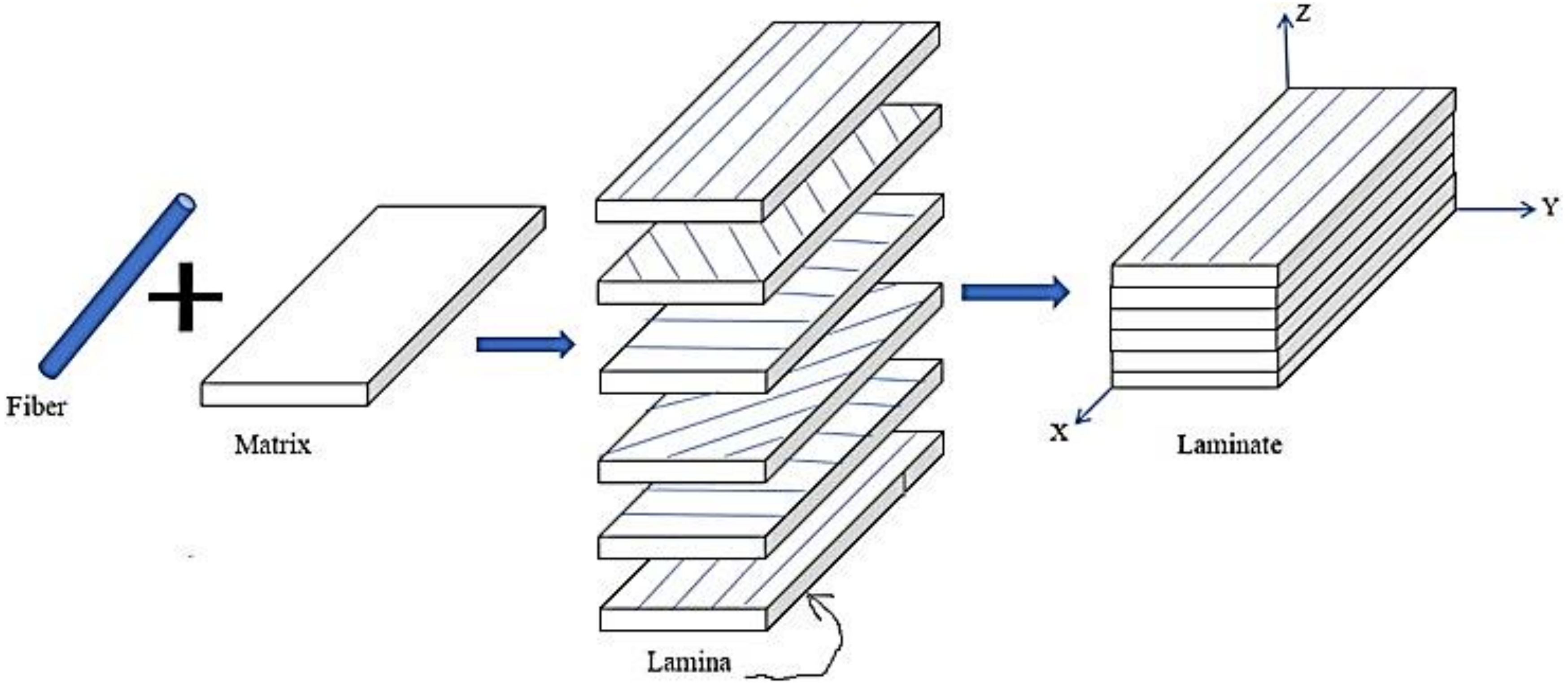
Summary

1. Review definitions of displacements, stresses, strains, and strain energy.
2. Develop stress–strain relationships for different types of materials.
3. Develop stress–strain relationships for a unidirectional (UD) lamina.
4. Find the engineering constants of a UD lamina in terms of the stiffness and compliance parameters of the lamina.
5. Develop stress–strain relationships, elastic moduli, strengths, and thermal and moisture expansion coefficients of an angle lamina based on those of a UD lamina.
6. Study strength failure theories for an angle lamina.

Contents

2.1. Review of Definitions

A lamina is a thin layer of a composite material that is generally of a thickness on the order of 0.005 in. (0.125 mm). A laminate is constructed by stacking a number of such laminae in the direction of the lamina thickness.



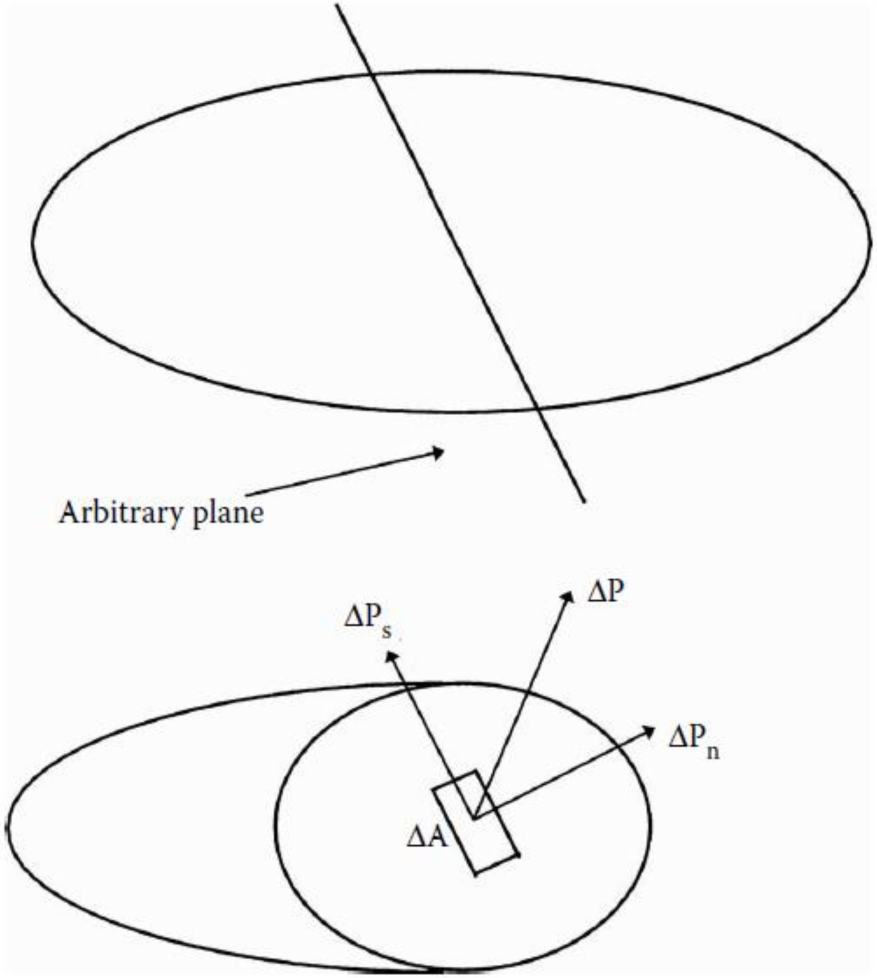
Stresses

Normal stress

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A},$$

Shear stress

$$\tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A},$$



Stresses on infinitesimal area on an arbitrary plane

Stresses

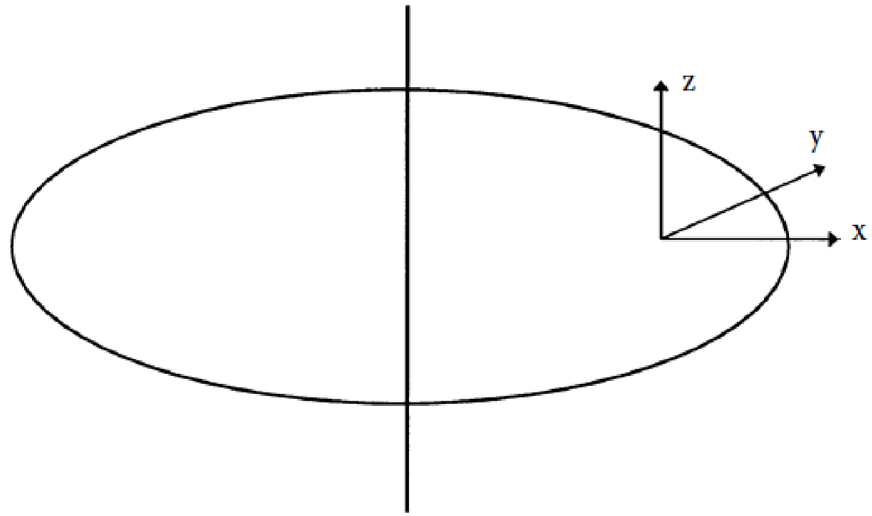
Normal stress

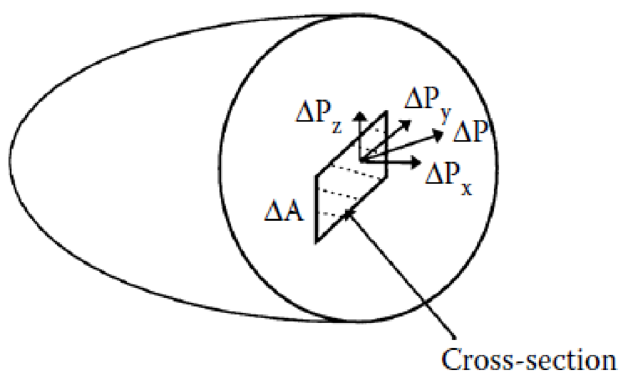
$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A},$$

Shear stresses

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A},$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}$$





Forces on an infinitesimal area on the y-z plane

Stresses

Normal stresses:

$$\sigma_x, \sigma_y, \sigma_z$$

Shear stresses:

$$\tau_{xy}, \tau_{yx}, \tau_{yz},$$

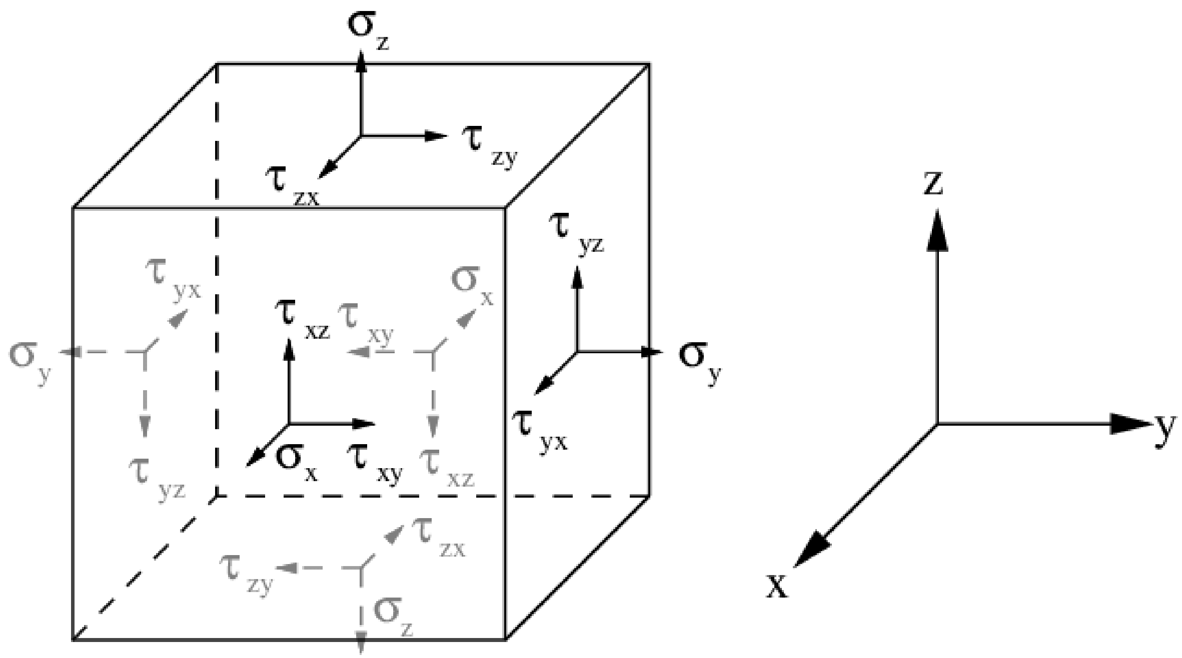
$$\tau_{zy}, \tau_{zx}, \tau_{xz}$$

Static equilibrium

$$\tau_{xy} = \tau_{yx},$$

$$\tau_{yz} = \tau_{zy},$$

$$\tau_{zx} = \tau_{xz}$$



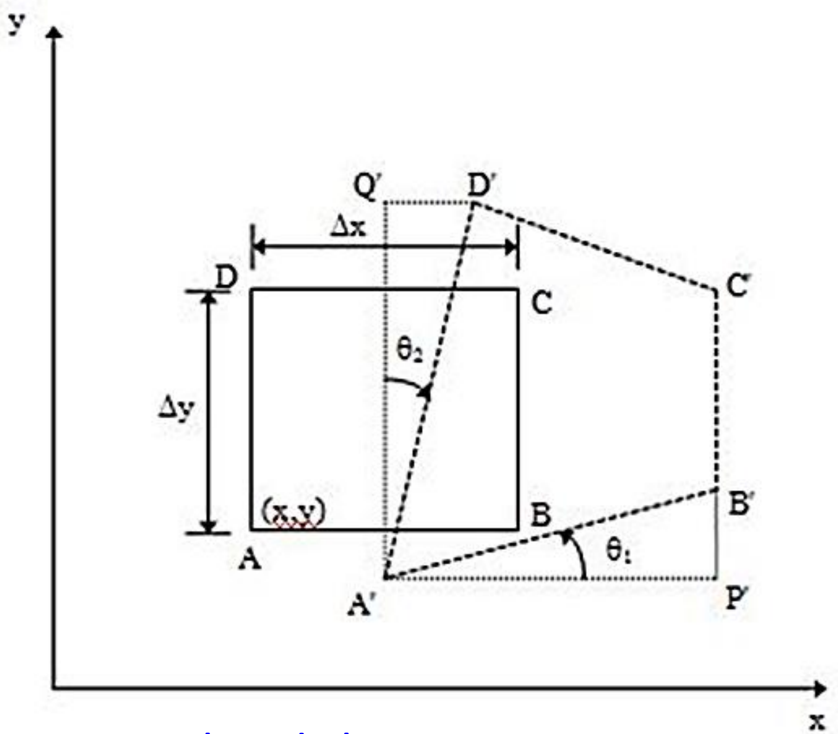
Stresses acting on a volume element
(an infinitesimal cuboid)

Displacements

$u = u(x,y,z)$: displacement in x-direction at point (x,y,z),

$v = v(x,y,z)$: displacement in y-direction at point (x,y,z),

$w = w(x,y,z)$: displacement in z-direction at point (x,y,z)



Normal and shear strains on an infinitesimal area in the x-y plane

Strains

Normal strains

$$\varepsilon_x, \varepsilon_y, \varepsilon_z$$

Shear strains

$$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$$

$$\varepsilon_x = \lim_{AB\to 0} \frac{A'B' - AB}{AB}, \quad \varepsilon_y = \lim_{AD\to 0} \frac{A'D' - AD}{AD},$$

$$\gamma_{xy} = \theta_1 + \theta_2$$

Normal strain: $$\varepsilon_x = \lim_{AB\to 0} \frac{A'B' - AB}{AB},$$

Where: $AB = \Delta x,$

$$A'B' = \sqrt{(A'P')^2 + (B'P')^2} = \sqrt{[\Delta x + u(x+\Delta x, y) - u(x,y)]^2 + [v(x+\Delta x, y) - v(x,y)]^2},$$

Substituting

$$\varepsilon_x = \lim_{\Delta x\to 0} \left\{ \left[1 + \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right]^2 + \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]^2 \right\}^{1/2} - 1$$

$$\longrightarrow \varepsilon_x = \left[\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \right]^{1/2} - 1$$

$$\frac{\partial u}{\partial x} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial x} << 1 \longrightarrow$$

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}}$$

Normal strain: $$\varepsilon_y = \lim_{AD \to 0} \frac{A'D' - AD}{AD},$$

Where: $AD = \Delta y$

$$A'D' = \sqrt{(A'Q')^2 + (Q'D')^2} = \sqrt{[\Delta y + v(x,y+\Delta,) - v(x,y)]^2 + [u(x,y+\Delta,) - u(x,y)]^2},$$

Similarly, substituting

$$\varepsilon_y = \lim_{\Delta y \to 0} \left\{ \left[1 + \frac{v(x,y+\Delta y) - v(x,y)\}}{\Delta y} \right]^2 + \left[\frac{u(x,y+\Delta y) - u(x,y)}{\Delta y} \right]^2 \right\}^{1/2} - 1$$

$$\longrightarrow \quad \varepsilon_y = \left[\left(1 + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{1/2} - 1$$

$$\frac{\partial u}{\partial y} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial y} << 1 \longrightarrow$$

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}}$$

Shear strain: $$\gamma_{xy} = \theta_1 + \theta_2$$

Where:

$$\theta_1 = \lim_{AB\to 0} \frac{P'B'}{A'P'}, \qquad \theta_2 = \lim_{AD\to 0} \frac{Q'D'}{A'Q'},$$

$$P'B' = v(x+\Delta x, y) - v(x,y),$$

$$A'P' = u(x+\Delta x, y) + \Delta x - u(x,y)$$

$$Q'D' = u(x, y+\Delta y) - u(x,y),$$

$$A'Q' = v(x, y+\Delta y) + \Delta y - v(x,y)$$

Substituting

$$\gamma_{xy} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\dfrac{v(x+\Delta x,y)-v(x,y)}{\Delta x}}{\dfrac{u(x+\Delta x,y)+\Delta x-u(x,y)}{\Delta x}} + \frac{\dfrac{u(x,y+\Delta y)-u(x,y)}{\Delta y}}{\dfrac{v(x,y+\Delta y)+\Delta y-v(x,y)}{\Delta y}}$$

$$\gamma_{xy} = \frac{\dfrac{\partial v}{\partial x}}{1+\dfrac{\partial u}{\partial x}} + \frac{\dfrac{\partial u}{\partial y}}{1+\dfrac{\partial v}{\partial y}}$$

$$\longrightarrow \frac{\partial u}{\partial x} << 1$$

$$\longrightarrow \frac{\partial v}{\partial y} << 1$$

$$\longrightarrow \boxed{\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$

2.1. Review of Definitions

Strains

Similarly $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$

Normal strains

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}}$$

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}}$$

$$\boxed{\varepsilon_z = \frac{\partial w}{\partial z}}$$

Shear strains

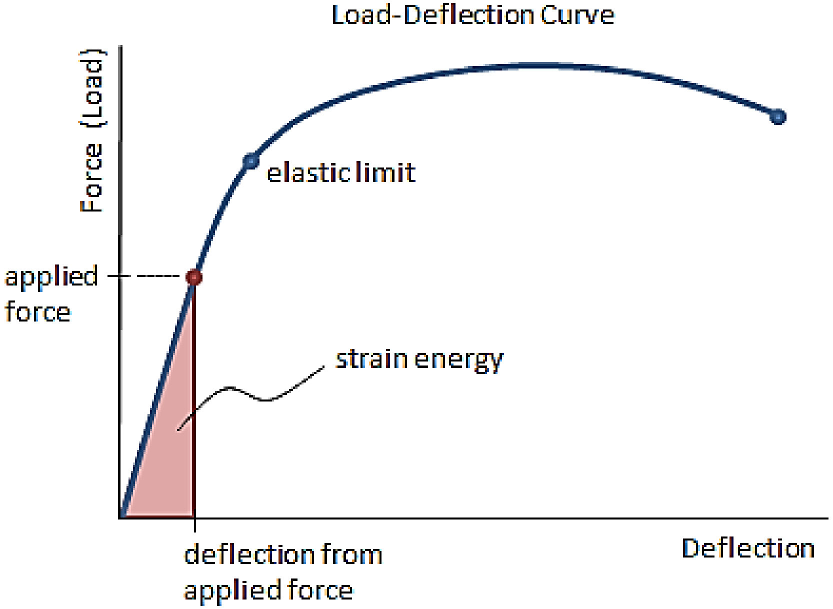
$$\boxed{\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$

$$\boxed{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},}$$

$$\boxed{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},}$$

Strain Energy

Energy is defined as the capacity to do work. In a deformable elastic body under external loads, the work done by external loads is stored as recoverable strain energy. For linearly elastic materials, strain energy is:



$$W = \frac{1}{2} V \sigma \varepsilon;$$

σ is stress, ε is strain, V is volume

The strain energy stored in the body per unit volume is then defined as

$$W = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx})$$

$$\text{or } W = \frac{1}{2}\sum_{i=1}^{6} \sigma_{\mathrm{i}} \varepsilon_{\mathrm{i}}$$

2.2. Hooke's Law for Different Types of Materials

Generalized Hooke's Law: the most general linear relation among all the components of the stress and strain tensor:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}\ ;\quad i,j,k,l = 1..3$$

C_{ijkl} are the components of the fourth-order stiffness tensor of material properties or Elastic moduli. The fourth-order stiffness tensor has 81 and 16 components for 3D and 2D problems, respectively.

Symmetries of the stiffness tensor:

The stiffness tensor has the following minor symmetries which result from the symmetry of the stress tensors.

$$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl}$$

This reduces the number of material constants from 81 = 3 × 3 × 3 × 3 → 54 = 6 × 3 × 3. In a similar fashion we can make use of the symmetry of the strain tensors:

$$\varepsilon_{ij} = \varepsilon_{ji} \Rightarrow C_{ijkl} = C_{ijlk}$$

This further reduces the number of material constants to 36 = 6 × 6.

2.2. Hooke's Law for Different Types of Materials

Engineering or Voigt notation

Since the tensor notation is already lost in the matrix notation, therefore we might give indices to all the components that make more sense for matrix operation. The general stress–strain relation for a 3D body in a 1–2–3 orthogonal Cartesian coordinate system is:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} \qquad (2.2.1)$$

Stiffness matrix [C] has 36 constants

OR

Compliance matrix [S] has 36 constants

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} \qquad (2.2.2)$$

2.2. Hooke's Law for Different Types of Materials

It can be shown that the 36 constants in Equation (2.2.1) actually reduce to 21 constants due to the symmetry of the stiffness matrix [C] as follows:
The stress–strain relationship (2.2.1) can also be written as

$$\sigma_i = \sum_{j=1}^{6} C_{ij}\varepsilon_j \ , \quad i = 1..6 \qquad [\sigma] = [C][\varepsilon]$$

where, in a contracted notation,

$$\begin{cases} \sigma_4 = \tau_{23}, & \sigma_5 = \tau_{31}, & \sigma_6 = \tau_{12}, \\ \varepsilon_4 = \gamma_{23}, & \varepsilon_5 = \gamma_{31}, & \varepsilon_6 = \gamma_{12}, \end{cases}$$

Substituting Hooke's law to the strain energy

$$W = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} C_{ij}\varepsilon_j\varepsilon_i \Rightarrow \frac{\partial W}{\partial\varepsilon_i\partial\varepsilon_j} = C_{ij} \quad \text{and} \quad \frac{\partial W}{\partial\varepsilon_j\partial\varepsilon_i} = C_{ji}$$

Because the differentiation does not necessarily need to be in either order,

$$C_{ij} = C_{ji}$$

Therefore, only 21 independent elastic constants are in the general stiffness matrix [C] of Equation (2.2.1). This also implies that only 21 independent constants are in the general compliance matrix [S] of Equation (2.2.2).

$$\varepsilon_i = \sum_{j=1}^{6} S_{ij}\sigma_j \ , \quad i = 1..6 \qquad [\varepsilon] = [S][\sigma]$$

Anisotropic Materials

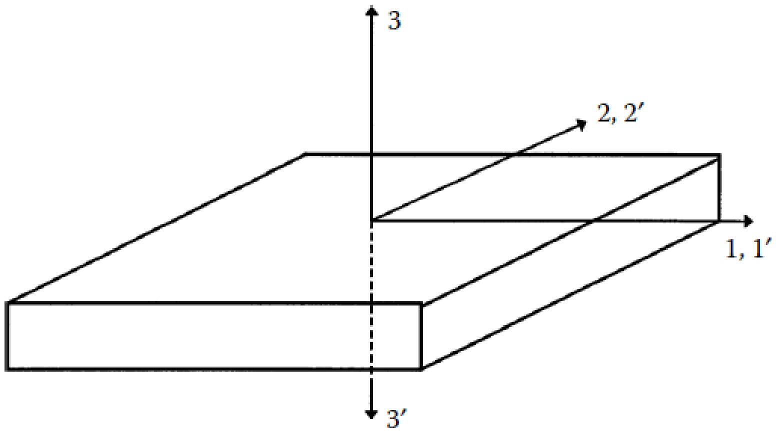
- The material that has 21 independent elastic constants at a point is called an anisotropic material.
- However, many natural and synthetic materials do possess material symmetry — that is, elastic properties are identical in directions of symmetry because symmetry is present in the internal structure. Fortunately, this symmetry reduces the number of the independent elastic constants by zeroing out or relating some of the constants within the 6 × 6 stiffness [C] and 6 × 6 compliance [S] matrices. This simplifies the Hooke’s law relationships for various types of elastic symmetry.
- Independent elastic constants for various types of materials such as monoclinic materials, orthotropic materials, and transversely isotropic, and isotropic materials are presented as follows.

Monoclinic Materials

If, in one plane of material symmetry (see figure below), for example, direction 3 is normal to the plane of material symmetry, then the stiffness matrix reduces to

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

$$C_{14}=0,\ C_{15}=0,\ C_{24}=0,\ C_{25}=0,$$
$$C_{34}=0,\ C_{35}=0,\ C_{46}=0,\ C_{56}=0$$



Transformation of coordinate axes for 1-2 plane of symmetry for a monoclinic material.

The direction perpendicular to the plane of symmetry is called the principal direction. Note that there are 13 independent elastic constants.

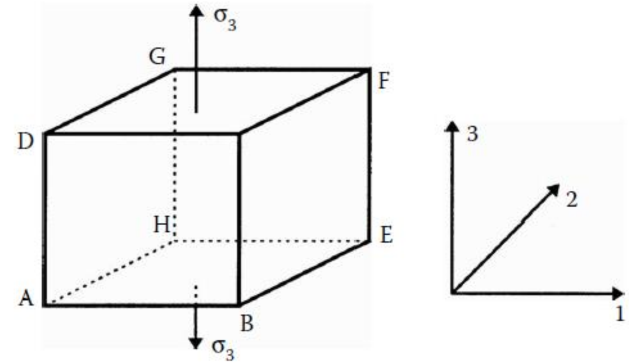
Minerals that form in the monoclinic system include azurite, brazilianite, crocoite, datolite, diopside, jadeite, lazulite, malachite, orthoclase feldspars.

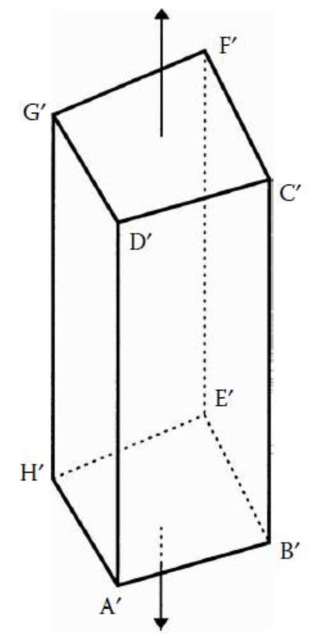
2.2. Hooke's Law for Different Types of Materials

Monoclinic Materials

The compliance matrix correspondingly reduces to

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}$$





Deformation of a cubic element made of monoclinic material.

Apply a normal stress, σ_3, to the element. Then using the Hooke's law Equation (2.2.2) and the compliance matrix [S] for the monoclinic material, one gets

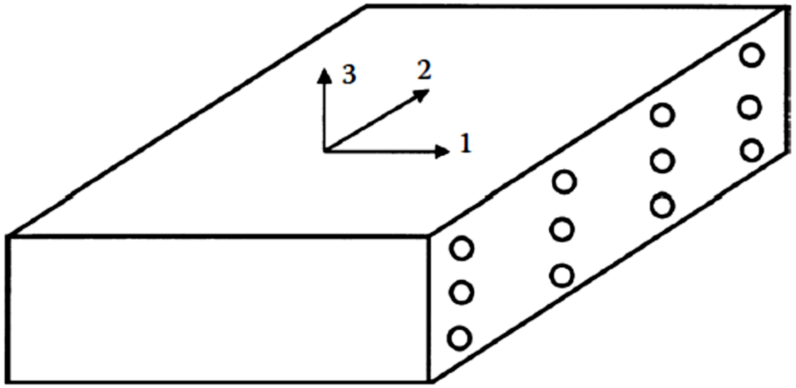
$$\varepsilon_1 = S_{13}\sigma_3; \quad \varepsilon_2 = S_{23}\sigma_3; \quad \varepsilon_3 = S_{33}\sigma_3;$$

$$\gamma_{23} = 0; \quad \gamma_{31} = 0;; \quad \gamma_{12} = S_{36}\sigma_3.$$

Orthotropic Materials

If a material has three mutually perpendicular planes of material symmetry, then the stiffness matrix is given by

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$



An orthotropic material include a single lamina of continuous fiber arranged in a rectangular array

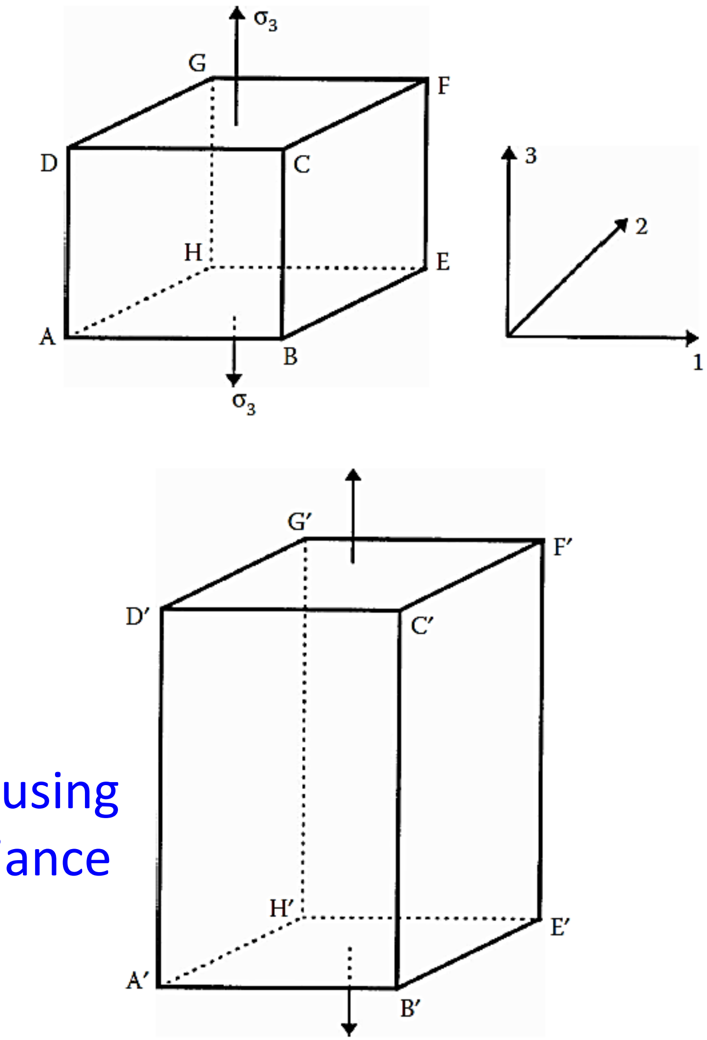
$$C_{14}=C_{15}=C_{16}=C_{24}=C_{26}=C_{34}=C_{35}=C_{36}=C_{55}=C_{46}=C_{56}=0$$

Three mutually perpendicular planes of material symmetry also imply three mutually perpendicular planes of elastic symmetry. Note that 9 independent elastic constants are present.

Orthotropic Materials

The compliance matrix reduces to

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$



Deformation of a cubic element made of orthotropic material.

Apply a normal stress, σ_3, to the element. Then using the Hooke's law Equation (2.2.2) and the compliance matrix [S] for the monoclinic material, one gets

$$\varepsilon_1 = S_{13}\sigma_3; \quad \varepsilon_2 = S_{23}\sigma_3; \quad \varepsilon_3 = S_{33}\sigma_3;$$

$$\gamma_{23} = 0; \quad \gamma_{31} = 0; \quad \gamma_{12} = 0.$$

Orthotropic Materials

Engineering constants have been defined as follows:

- ✓ Three Young's moduli, E_1, E_2, and E_3.
- ✓ Six Poisson's ratios, ν_{12}, ν_{13}, ν_{21}, ν_{23}, ν_{31}, and ν_{32}, two for each plane.
- ✓ Three shear moduli, G_{23}, G_{31}, and G_{12}, one for each plane.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

However, the six Poisson's ratios are not independent of each other.

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}; \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}.$$

Orthotropic Materials

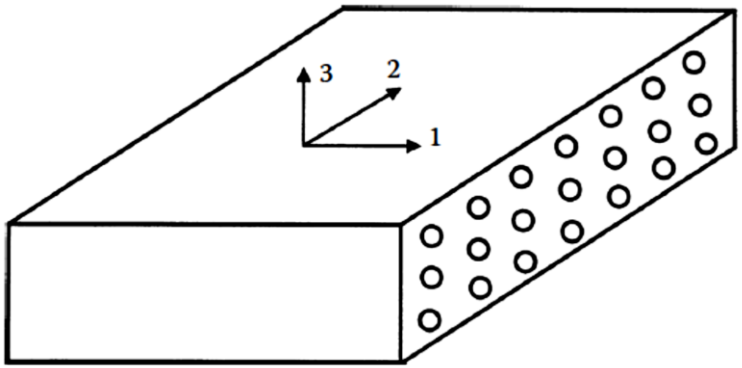
$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

where $\Delta = \left(1-\nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{31}\right) / \left(E_1E_2E_3\right)$

Transversely Isotropic Material

Consider a plane of material isotropy in one of the planes of an orthotropic body. If direction 1 is normal to that plane (2–3) of isotropy, then the stiffness matrix is given by

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$



A unidirectional lamina as a transversely isotropic material with fibers arranged in a square array.

Note the five independent elastic constants. An example of this is a thin unidirectional lamina in which the fibers are arranged in a square array or a hexagonal array.

2.2. Hooke's Law for Different Types of Materials

Transversely Isotropic Material

The compliance matrix reduces to

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{12}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

Note that 5 independent elastic constants (S_{11}, S_{12}, S_{22}, S_{23}, S_{55}) corresponding to (E_1, E_2, ν_{12}, G_{23}, G_{12}) are present.

$$G_{23}=\frac{E_2}{2(1+\nu_{23})}$$

Isotropic Material

If all planes in an orthotropic body are identical, it is an isotropic material; then, the stiffness matrix is given by

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

$$[\sigma]=[C][\varepsilon] \qquad G=\frac{E}{2(1+\nu)}$$

This also implies infinite principal planes of symmetry. Note that 2 independent elastic constants corresponding to (E, ν) are present.

2.2. Hooke's Law for Different Types of Materials

Isotropic Material

The compliance matrix reduces to

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix}$$

$$[\varepsilon]=[S][\sigma] \qquad G=\frac{E}{2(1+\nu)}$$

Note that 2 independent elastic constants (S_{11}, S_{12}) corresponding to (E, ν) are present.

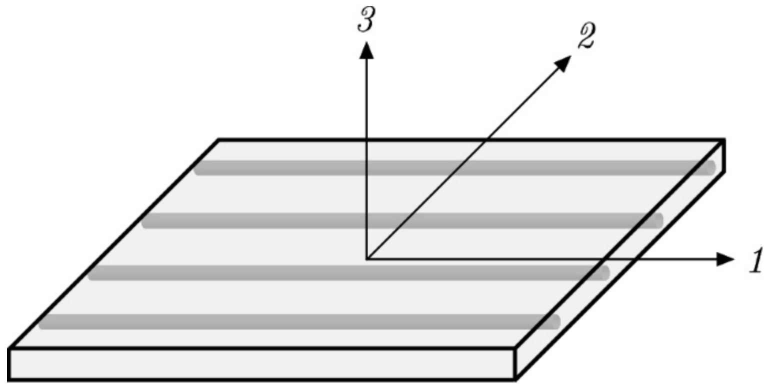
2.2. Hooke's Law for Different Types of Materials

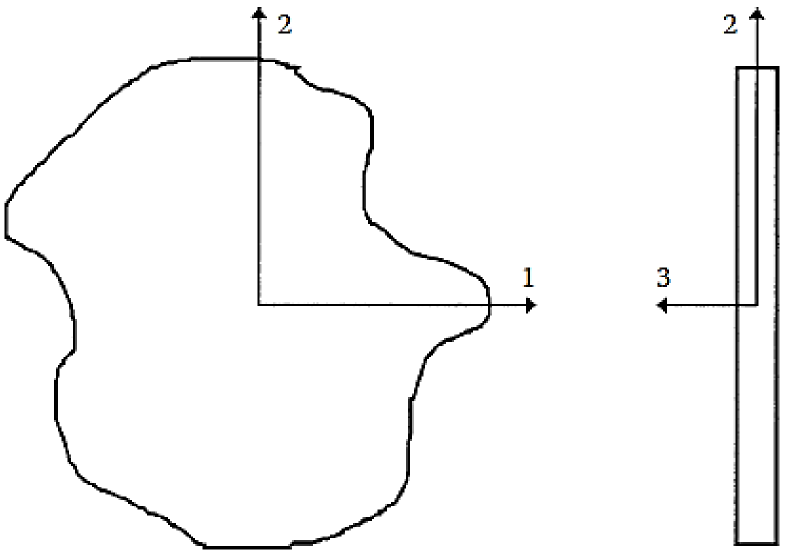
Summarize the number of nonzero constants and independent elastic constants for various types of materials:

Material Type	Three dimensional		Two dimensional	
	Number of nonzero constants	**Independent Elastic Constants**	Number of nonzero constants	**Number of independent constants**
Anisotropic	36	21	9	6
Orthotropic	12	9	5	4
Transversely Isotropic	12	5	5	4
Isotropic	12	2	5	2

Plane Stress Assumption

- ✓ A thin plate is a prismatic member having a small thickness, and it is the case for a typical lamina. If a plate is thin and there are no out-of-plane loads, it can be considered to be under plane stress.
- ✓ A lamina is thin and, if no out-of-plane loads are applied, one can assume that it is under plane stress.
- ✓ This assumption then reduces the three-dimensional stress–strain equations to two-dimensional stress–strain equations.



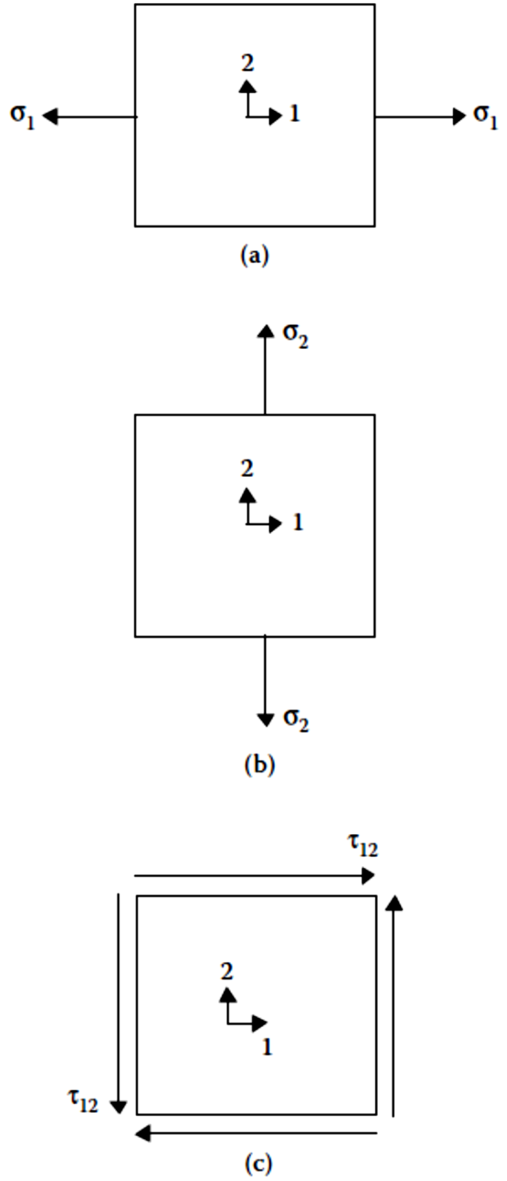


Plane stress conditions for a thin plate.

Reduction of Hooke's Law in 3D to 2D

- A unidirectional lamina falls under the orthotropic material category.
- If the upper and lower surfaces of the lamina are free from external loads, then $\sigma_3 = 0$, $\tau_{31} = 0$, and $\tau_{23} = 0$.
- The strain–stress relationship for orthotropic plane stress problem can then be written as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$



2.3. Hooke's Law for a 2D Unidirectional Lamina

- Inverting above equation gives the stress–strain relationship as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

where Q_{ij} are the reduced stiffness elements, which are related to the compliance elements as

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}, \quad Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}$$

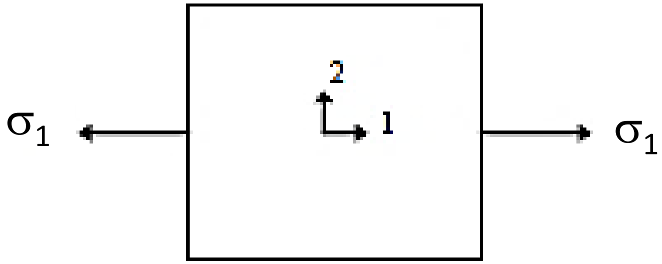
$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}, \quad Q_{66} = \frac{1}{S_{66}}$$

2.3. Hooke's Law for a 2D Unidirectional Lamina

Relationship of Compliance and Stiffness Matrix to Engineering Elastic Constants of a Lamina

Apply pure axial load in direction 1

$$\sigma_1 \neq 0, \; \sigma_2 = 0, \; \tau_{12} = 0$$



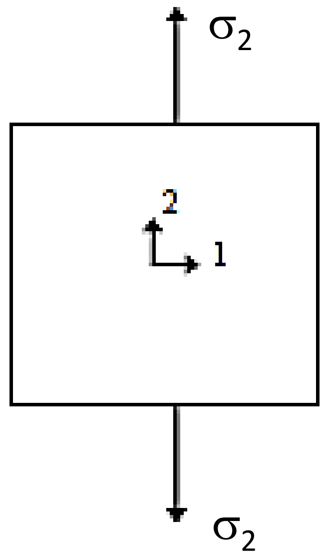
$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \cancel{\sigma_2} \\ \cancel{\tau_{12}} \end{bmatrix} \longrightarrow \begin{array}{l} \varepsilon_1 = S_{11}\sigma_1 \\ \varepsilon_2 = S_{12}\sigma_1 \\ \gamma_{12} = 0 \end{array}$$

Hooke's law $E_1 \equiv \dfrac{\sigma_1}{\varepsilon_1} = \dfrac{1}{S_{11}} \longrightarrow \boxed{S_{11} = \dfrac{1}{E_1}}$

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}} \longrightarrow \boxed{S_{12} = -\frac{\nu_{12}}{E_1}}$$

2.3. Hooke's Law for a 2D Unidirectional Lamina

Apply a pure axial load in direction 2



$$\sigma_1 = 0,\ \ \sigma_2 \neq 0,\ \ \tau_{12} = 0$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \cancel{\sigma_1} \\ \sigma_2 \\ \cancel{\tau_{12}} \end{bmatrix} \longrightarrow \begin{aligned} \varepsilon_1 &= S_{12}\sigma_2 \\ \varepsilon_2 &= S_{22}\sigma_2 \\ \gamma_{12} &= 0 \end{aligned}$$

$$E_2 \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}} \longrightarrow \boxed{S_{22} = \frac{1}{E_2}}$$

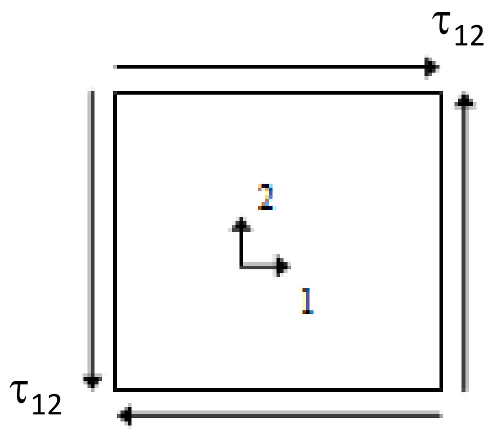
$$\nu_{21} \equiv -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}} \longrightarrow \boxed{S_{12} = -\frac{\nu_{21}}{E_2}}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} \qquad S_{12} = -\frac{\nu_{21}}{E_2}$$

$$\boxed{\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}}$$

2.3. Hooke's Law for a 2D Unidirectional Lamina

Apply a pure shear load in plane 1-2



$$\sigma_1 = 0,\ \sigma_2 = 0,\ \tau_{12} \neq 0$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \cancel{\sigma_1} \\ \cancel{\sigma_2} \\ \tau_{12} \end{bmatrix} \longrightarrow \begin{array}{l} \varepsilon_1 = 0 \\ \varepsilon_2 = 0 \\ \gamma_{12} = S_{66}\tau_{12} \end{array}$$

$$G_{12} \equiv \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}} \longrightarrow \boxed{S_{66} = \frac{1}{G_{12}}}$$

Compliance Matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Reduced Stiffness Matrix

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{66} = \frac{1}{S_{66}}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1 - \nu_{21}\nu_{12}} & \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} & 0 \\ \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} & \frac{E_2}{1 - \nu_{21}\nu_{12}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

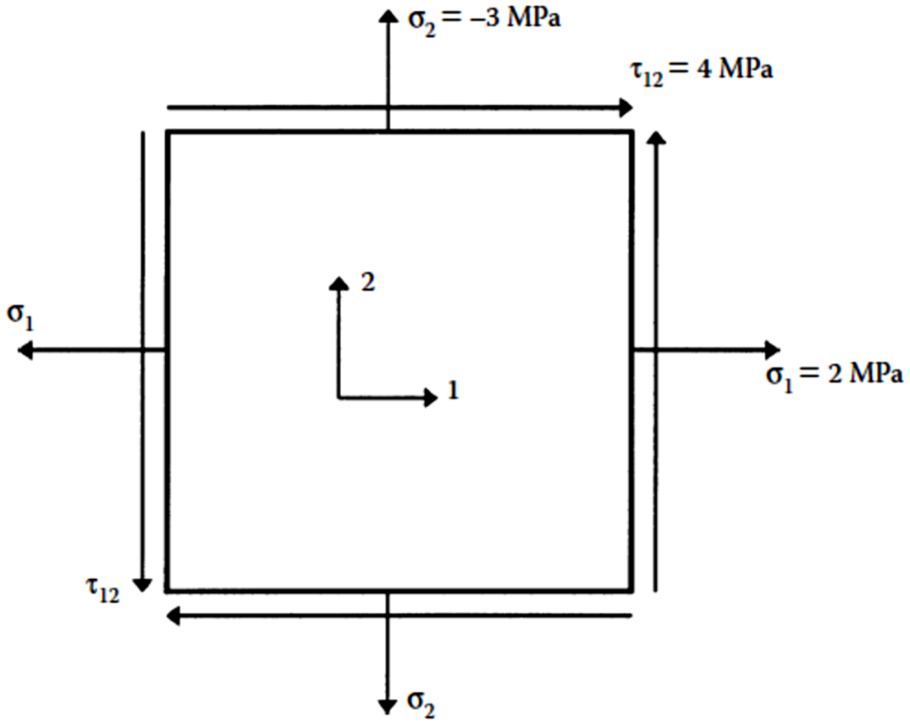
Example 2.1

For a graphite/epoxy unidirectional lamina, find the following

1. Compliance matrix
2. Minor Poisson's ratio
3. Reduced stiffness matrix
4. Strains in the 1–2 coordinate system if the applied stresses (Figure 2.1) are

$$\sigma_1 = 2\text{ MPa}, \sigma_2 = -3\text{ MPa}, \tau_{12} = 4\text{ MPa}.$$

Use the properties of unidirectional graphite/epoxy lamina from Table 2.1 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006).



Applied stresses in a unidirectional lamina in Example 2.1.

Solution

From Table 2.1 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006), the engineering elastic constants of the unidirectional graphite/epoxy lamina are.

$E_1 = 181$ GPa, $E_2 = 10.3$ GPa, $\nu_{12} = 0.28$, $G_{12} = 7.17$ GPa.

1. The compliance matrix elements are

$$S_{11} = \frac{1}{E_1} = \frac{1}{181\times 10^9} = 0.5525\times 10^{-11}\,Pa^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181\times 10^9} = -0.1547\times 10^{-11}\,Pa^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3\times 10^9} = 0.9709\times 10^{-10}\,Pa^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17\times 10^9} = 0.1395\times 10^{-9}\,Pa^{-1}$$

Examples

2. The minor Poisson's ratio is

$$\nu_{21} = \frac{\nu_{12}}{E_1} E_2 = \frac{0.28}{181 \times 10^9} \times \left(10.3 \times 10^9\right) = 0.01593$$

3. The reduced stiffness matrix [Q] elements are

$$Q_{11} = \frac{E_1}{1 - \nu_{21} \nu_{12}}$$

$$Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{21} \nu_{12}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21} \nu_{12}}$$

$$Q_{66} = G_{12}$$

$$[Q] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9) \text{ Pa}$$

Note that the reduced stiffness matrix [*Q*] could also be obtained by inverting the compliance matrix [S].

Examples

4. The strains in the 1–2 coordinate system are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 15.69 \\ -294.4 \\ 557.9 \end{bmatrix} \left(10^{-6}\right)$$

Thus, the strains in the local axes are

$$\varepsilon_1 = 15.69 \ \frac{\mu m}{m}$$

$$\varepsilon_2 = -294.4 \ \frac{\mu m}{m}$$

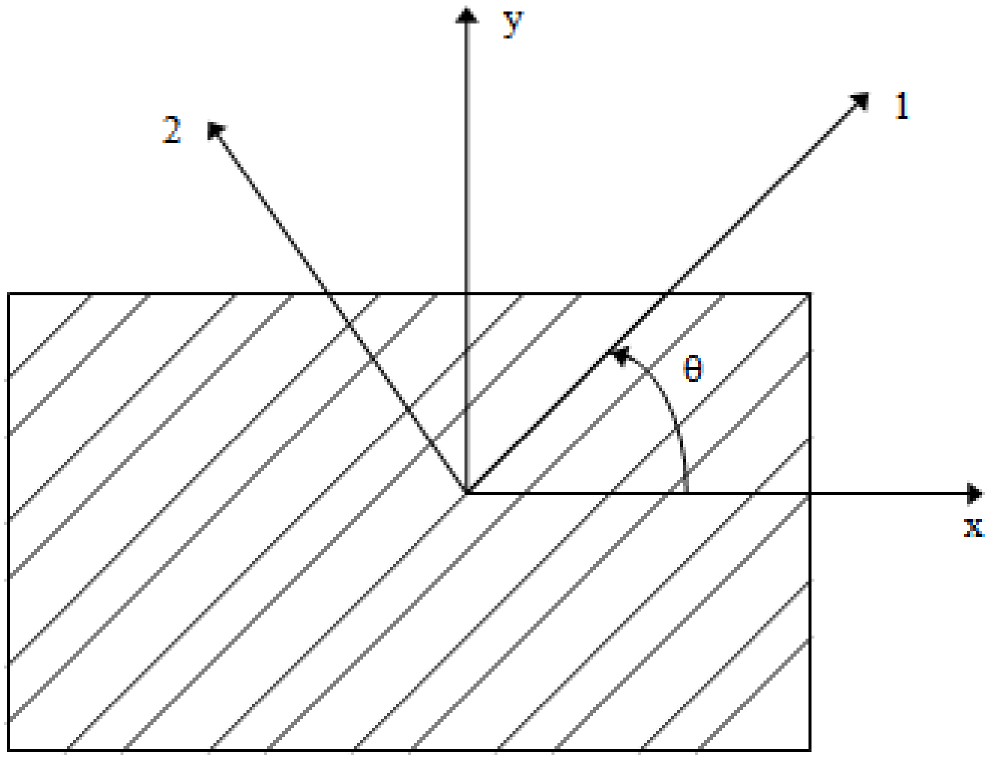
$$\gamma_{12} = 557.9 \ \frac{\mu m}{m}$$

2.4. Hooke's Law for a 2D Angle Lamina

A laminate does not consist only of unidirectional laminae because of their low stiffness and strength properties in the transverse direction. Therefore, in most laminates, some laminae are placed at an angle. It is necessary to develop the stress–strain relationship for an angle lamina.

The coordinate system used for showing an angle lamina is as given in the figure.

- ✓ The axes in the 1–2 coordinate system are called the local axes or the material axes.
- ✓ The axes in the x–y coordinate system are called the global axes or the off-axes.



Local and global axes of an angle lamina

Transformation of Stresses

The global and local stresses in an angle lamina are related to each other through the lamina angle, θ:

Notation
$c = \cos(\theta)$, $s = \sin(\theta)$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

where $[T]$ is called the transformation matrix.

Transformation of Strains

The global and local strains are also related through the transformation matrix:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix}$$

Expanding Global Strain-Local Strain Relationship

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} [T]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

where $[R]$ is the Reuter matrix and is defined as

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.4. Hooke's Law for a 2D Angle Lamina

Global Stress and Strain Relationship

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1}[S] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1}[S]\,[T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}}$$

$\left[\bar{S}\right] = [R][T]^{-1}[R]^{-1}[S][T]$ is the transformed compliance matrix.

Global Stress and Strain

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$[\bar{S}]$ is the transformed compliance matrix.

$$\bar{S}_{11} = S_{11} c^4 + (2S_{12} + S_{66}) s^2 c^2 + S_{22} s^4,$$

$$\bar{S}_{12} = S_{12} (s^4 + c^4) + (S_{11} + S_{22} - S_{66}) s^2 c^2,$$

$$\bar{S}_{22} = S_{11} s^4 + (2S_{12} + S_{66}) s^2 c^2 + S_{22} c^4,$$

$$\bar{S}_{16} = (2 S_{11} - 2 S_{12} - S_{66}) sc^3 - (2 S_{22} - 2 S_{12} - S_{66}) s^3 c,$$

$$\bar{S}_{26} = (2 S_{11} - 2 S_{12} - S_{66}) s^3 c - (2 S_{22} - 2 S_{12} - S_{66}) sc^3,$$

$$\bar{S}_{66} = 2 (2 S_{11} + 2 S_{22} - 4 S_{12} - S_{66}) s^2 c^2 + S_{66} (s^4 + c^4)$$

<u>Notation</u>

c = cos(θ), s = sin(θ)

Transformed Reduced Stiffness Matrix $[\bar{Q}]$

$$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1}$$

or $[\bar{Q}]=\mathrm{inv}(\bar{S})$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^4),$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)$$

Example 2.2

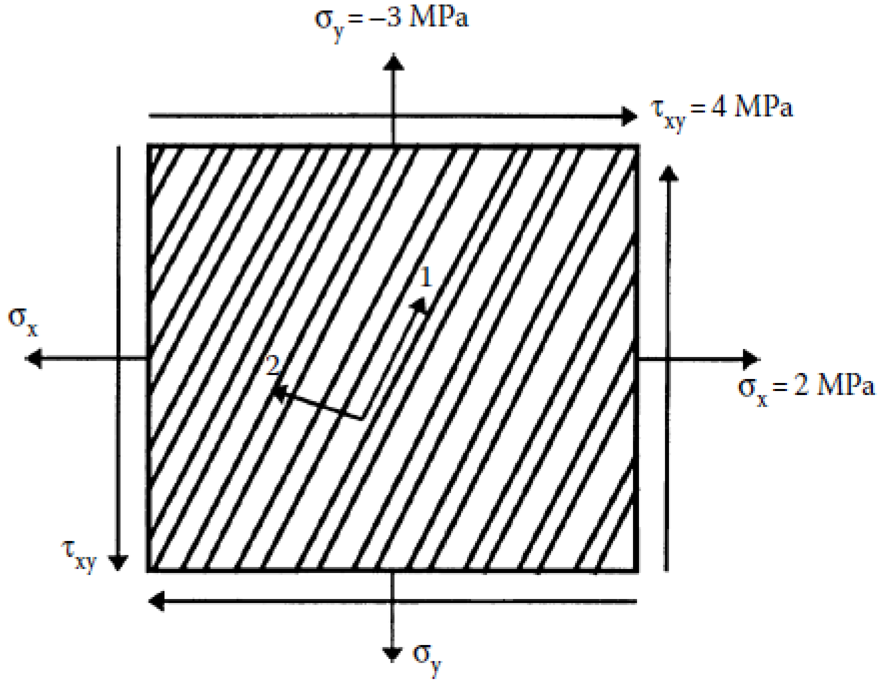
Find the following for a 60° angle lamina (see figure below) of graphite/epoxy. Use the properties of unidirectional graphite/epoxy lamina from Table 2.1 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006). Find the followings:

1. Transformed compliance matrix
2. Transformed reduced stiffness matrix

If the applied stress is

$$\sigma_x = 2 \text{ MPa}, \sigma_y = \text{-3 MPa}, \tau_{xy} = 4 \text{ MPa}.$$

3. Global strains
4. Local strains
5. Local stresses
6. Principal stresses
7. Maximum shear stress
8. Principal strains
9. Maximum shear strain



Examples

Solution

1. From Example 2.1

From Table 2.1

E_1 = 181 GPa,
E_2 = 10.3 GPa,
ν_{12} = 0.28,
G_{12} = 7.17 GPa.

$$S_{11} = \frac{1}{E_1} = \frac{1}{181\times10^9} = 0.5525\times10^{-11}\,Pa^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181\times10^9} = -0.1547\times10^{-11}\,Pa^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3\times10^9} = 0.9709\times10^{-10}\,Pa^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17\times10^9} = 0.1395\times10^{-9}\,Pa^{-1}$$

The elements of the transformed compliance matrix are given (slide 46)

$$\bar{S}_{11} = 0.8053\times10^{-10}\,Pa^{-1} \qquad \bar{S}_{12} = -0.7878\times10^{-11}\,Pa^{-1}$$

$$\bar{S}_{16} = -0.3234\times10^{-10}\,Pa^{-1} \qquad \bar{S}_{22} = 0.3475\times10^{-10}\,Pa^{-1}$$

$$\bar{S}_{26} = -0.4696\times10^{-10}\,Pa^{-1} \qquad \bar{S}_{66} = 0.1141\times10^{-9}\,Pa^{-1}$$

Examples

2. Invert the transformed compliance matrix [$\bar{S}$] to obtain the transformed reduced stiffness matrix [$\bar{Q}$]:

$$[\bar{Q}]=\mathrm{inv}(\bar{S}) = \begin{bmatrix} 0.2365 & 0.3246 & 0.2005 \\ 0.3246 & 0.1094 & 0.5419 \\ 0.2005 & 0.5419 & 0.3674 \end{bmatrix} (10^{11})\ \mathrm{Pa}$$

3. The global strains in the x–y plane are given

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 0.5534\times 10^{-4} \\ -0.3078\times 10^{-3} \\ 0.5328\times 10^{-3} \end{bmatrix}$$

Examples

4. The local strains in the lamina are following in which c = cos(60º), s = sin(60º)

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.1367 \times 10^{-4} \\ -0.2662 \times 10^{-3} \\ -0.5809 \times 10^{-3} \end{bmatrix}$$

5. The local stresses in the lamina are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 0.1714 \times 10^{7} \\ -0.2714 \times 10^{7} \\ -0.4165 \times 10^{7} \end{bmatrix} \text{ Pa}$$

Examples

6. The principal normal stresses are

$$\sigma_{\max/\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.217, \ \ -5.217\,MPa$$

The value of the angle at which the maximum normal stresses occur is

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = 29^{\mathrm{o}}$$

7. The maximum shear stress is given by

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.717\,MPa$$

The angle at which the maximum shear stress occurs is

$$\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) = 16^{\mathrm{o}}$$

8. The principal strains are given by

$$\varepsilon_{\max/\min} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 1.962\times 10^{-4},\ \ -4.486\times 10^{-4}$$

The value of the angle at which the maximum normal strains occur is

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) = 27.86^{\circ}$$

9. The maximum shear strain is given by

$$\gamma_{\max} = \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \gamma_{xy}^2} = 6.448\times 10^{-4}$$

The angle at which the maximum shear strain occurs is

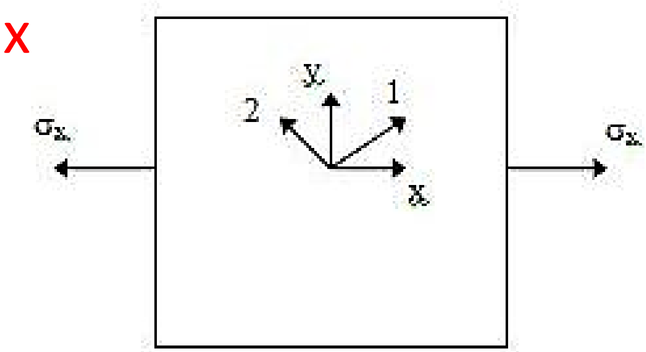
$$\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right) = -17.14^{\circ}$$

2.5. Engineering Constants of an Angle Lamina

The engineering constants for a unidirectional lamina were related to the compliance and stiffness matrices.

1. For finding the engineering elastic moduli in direction x, apply pure axial load in direction x

$$\sigma_x \neq 0, \sigma_y = 0, \tau_{xy} = 0$$



$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \cancel{\sigma_y} \\ \cancel{\tau_{xy}} \end{bmatrix} \Rightarrow \begin{aligned} \varepsilon_x &= \bar{S}_{11}\sigma_x \\ \varepsilon_y &= \bar{S}_{12}\sigma_x \\ \gamma_{xy} &= \bar{S}_{16}\sigma_x \end{aligned}$$

Application of stresses to find engineering constants of an angle lamina

$$E_x = \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\bar{S}_{11}} \qquad v_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}} \qquad \frac{1}{m_x} \equiv -\frac{\sigma_x}{\gamma_{xy} E_1} = -\frac{1}{\bar{S}_{16} E_1}$$

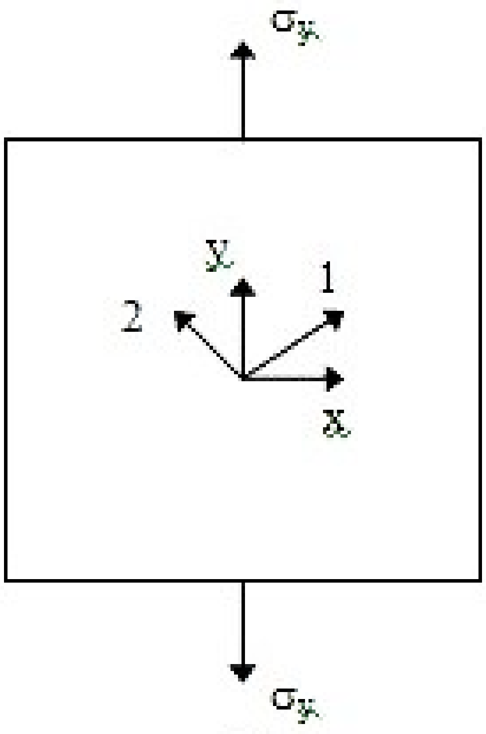
Shear coupling

2.5. Engineering Constants of an Angle Lamina

2. Similarly, apply pure axial load in direction y

$$\sigma_x = 0, \sigma_y \neq 0, \tau_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \cancelto{0}{\sigma_x} \\ \sigma_y \\ \cancelto{0}{\tau_{xy}} \end{bmatrix} \Rightarrow \begin{aligned} \varepsilon_x &= \bar{S}_{12}\sigma_y \\ \varepsilon_y &= \bar{S}_{22}\sigma_y \\ \gamma_{xy} &= \bar{S}_{26}\sigma_y \end{aligned}$$



$$E_y = \frac{\sigma_y}{\varepsilon_y} = \frac{1}{\bar{S}_{22}}$$

Shear coupling

$$\frac{1}{m_y} = -\frac{\sigma_y}{\gamma_{xy} E_1} = -\frac{1}{\bar{S}_{26} E_1}$$

$$\nu_{yx} = -\frac{\varepsilon_x}{\varepsilon_y} = -\frac{\bar{S}_{12}}{\bar{S}_{22}}$$

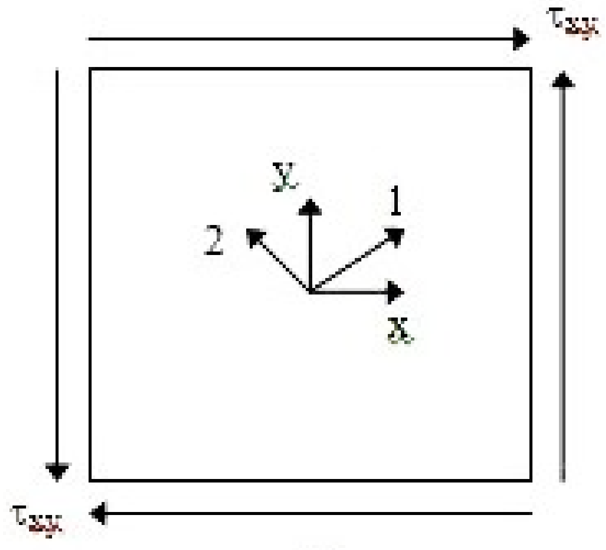
$$\boxed{\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}}$$

Application of stresses to find engineering constants of an angle lamina

3. Also, apply pure shear load in x-y plane

$$\sigma_x = 0, \sigma_y = 0, \tau_{xy} \neq 0$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \cancelto{0}{\sigma_x} \\ \cancelto{0}{\sigma_y} \\ \tau_{xy} \end{bmatrix} \Rightarrow \begin{aligned} \varepsilon_x &= \bar{S}_{16}\tau_{xy} \\ \varepsilon_y &= \bar{S}_{26}\tau_{xy} \\ \gamma_{xy} &= \bar{S}_{66}\tau_{xy} \end{aligned}$$



Application of stresses to find engineering constants of an angle lamina

$$\frac{1}{m_x} = -\frac{\tau_{xy}}{\varepsilon_x E_1} = -\frac{1}{\bar{S}_{16} E_1}$$

$$G_{xy} \equiv \frac{\tau_{xy}}{\gamma_{xy}} = \frac{1}{\bar{S}_{66}}$$

$$\frac{1}{m_y} = -\frac{\tau_{xy}}{\varepsilon_y E_1} = -\frac{1}{\bar{S}_{26} E_1}$$

Stress-Strain Relationships for Angle Lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\frac{1}{E_x} = \bar{S}_{11} \quad = S_{11} c^4 + \left(2 S_{12} + S_{66}\right) s^2 c^2 + S_{22} s^4$$

$$= \frac{1}{E_1} c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) s^2 c^2 + \frac{1}{E_2} s^4$$

2.5. Engineering Constants of an Angle Lamina

$$\frac{1}{E_y} = \bar{S}_{22} = S_{11} s^4 + (2S_{12} + S_{66}) c^2 s^2 + S_{22} c^4 = \frac{1}{E_1} s^4 + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) c^2 s^2 + \frac{1}{E_2} c^4$$

$$\frac{1}{G_{xy}} = \bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66}) s^2 c^2 + S_{66} (s^4 + c^4)$$

$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) s^2 c^2 + \frac{1}{G_{12}} (s^4 + c^4)$$

$$m_x = -\bar{S}_{16} E_1 = -E_1 \left[(2S_{11} - 2S_{12} - S_{66}) s c^3 - (2S_{22} - 2S_{12} - S_{66}) s^3 c \right]$$

$$= E_1 \left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) s c^3 + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) s^3 c \right]$$

$$m_y = -\bar{S}_{26} E_1 = -E_1 \left[(2S_{11} - 2S_{12} - S_{66}) s^3 c - (2S_{22} - 2S_{12} - S_{66}) s c^3 \right]$$

$$= E_1 \left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) s^3 c + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) s c^3 \right]$$

Example 2.3

Find the engineering constants of a 60° graphite/epoxy lamina. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1 (A.K. Kaw. Mechanics of Composite Materials. 2nd edition, Taylor & Francis, 2006).

Solution

From Example 2.2

$$\bar{S}_{11} = 0.8053 \times 10^{-10}\, Pa^{-1} \qquad \bar{S}_{12} = -0.7878 \times 10^{-11}\, Pa^{-1}$$

$$\bar{S}_{16} = -0.3234 \times 10^{-10}\, Pa^{-1} \qquad \bar{S}_{22} = 0.3475 \times 10^{-10}\, Pa^{-1}$$

$$\bar{S}_{26} = -0.4696 \times 10^{-10}\, Pa^{-1} \qquad \bar{S}_{66} = 0.1141 \times 10^{-9}\, Pa^{-1}$$

The engineering elastic moduli in direction x

$$E_x \equiv \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\bar{S}_{11}} = 12.42\ GPa$$

$$\nu_{xy} \equiv -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}} = 0.09783$$

$$\frac{1}{m_x} \equiv -\frac{\sigma_x}{\gamma_{xy} E_1} = -\frac{1}{\bar{S}_{16} E_1} \Rightarrow m_x = -\bar{S}_{16} E_1 = 5.854$$

The engineering elastic constants in direction y

$$E_y \equiv \frac{\sigma_y}{\varepsilon_y} = \frac{1}{\bar{S}_{22}} = 28.78 \text{ GPa}$$

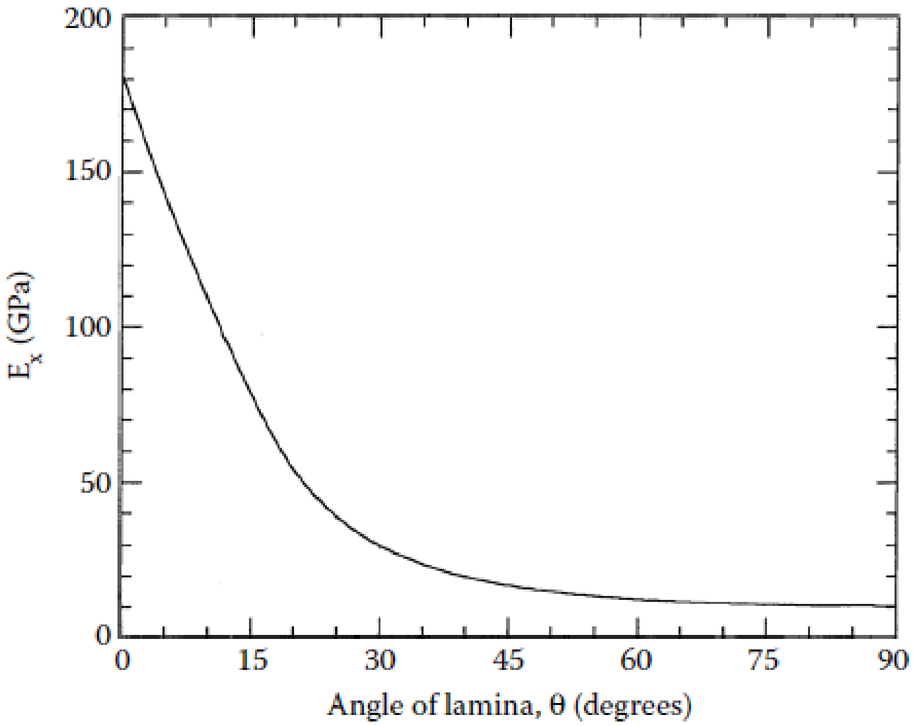
$$\frac{1}{m_y} \equiv -\frac{\sigma_y}{\gamma_{xy} E_1} = -\frac{1}{\bar{S}_{26} E_1} \Rightarrow m_y = -\bar{S}_{26} E_1 = 8.499$$

The engineering elastic constant in plane x - y

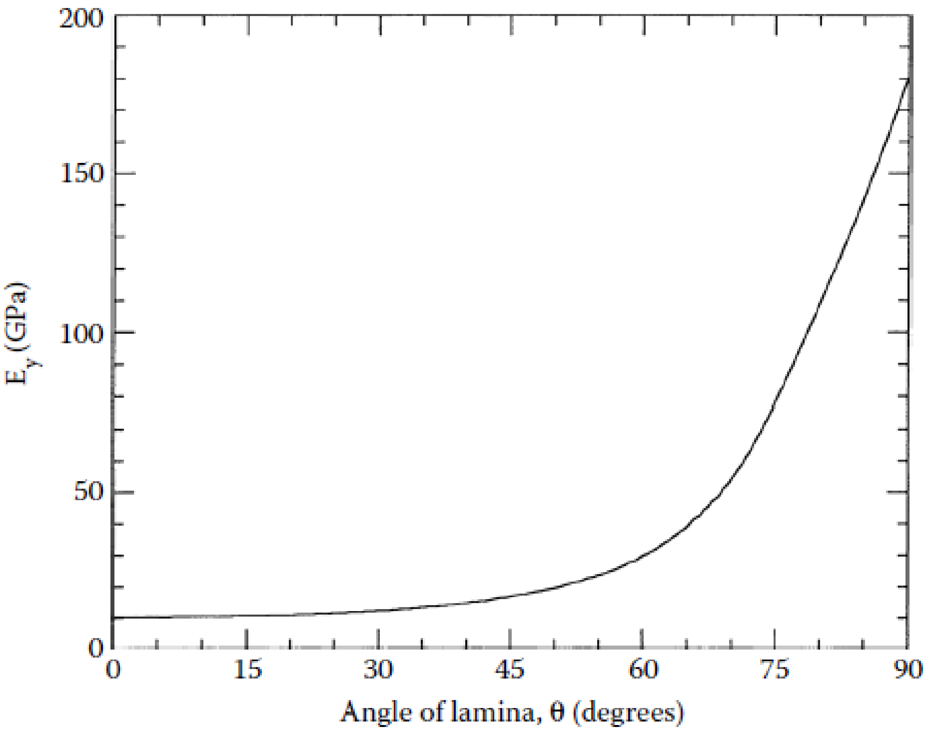
$$G_{xy} \equiv \frac{\tau_{xy}}{\gamma_{xy}} = \frac{1}{\bar{S}_{66}} = 8.761 \text{ GPa}$$

Examples

Elastic modulus in direction-x as a function of angle of lamina for a graphite/epoxy lamina.



Elastic modulus in direction-y as a function of angle of lamina for a graphite/epoxy lamina.

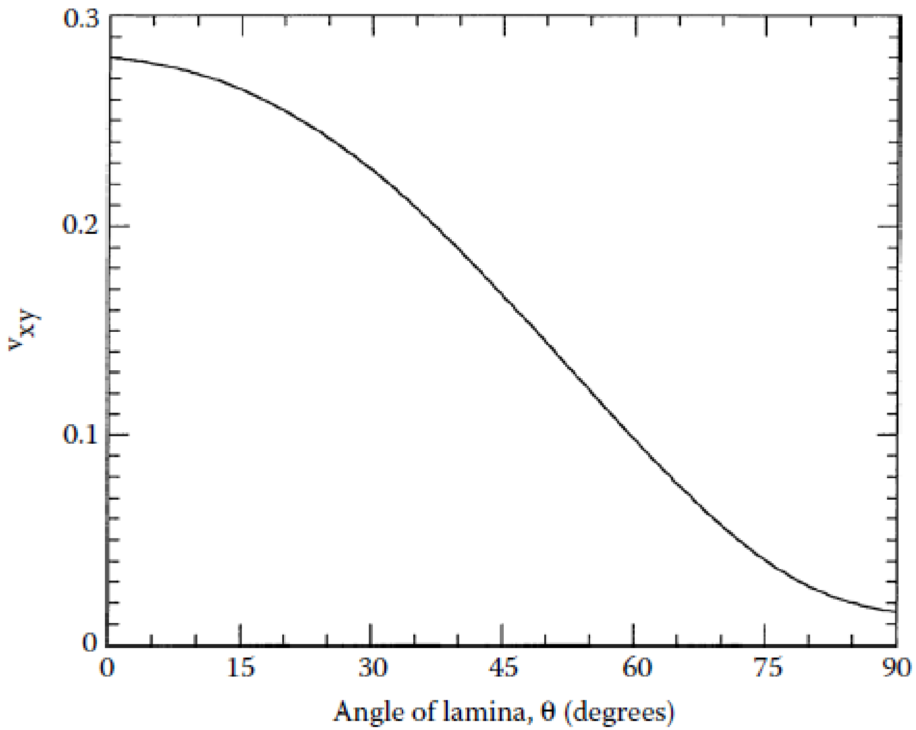


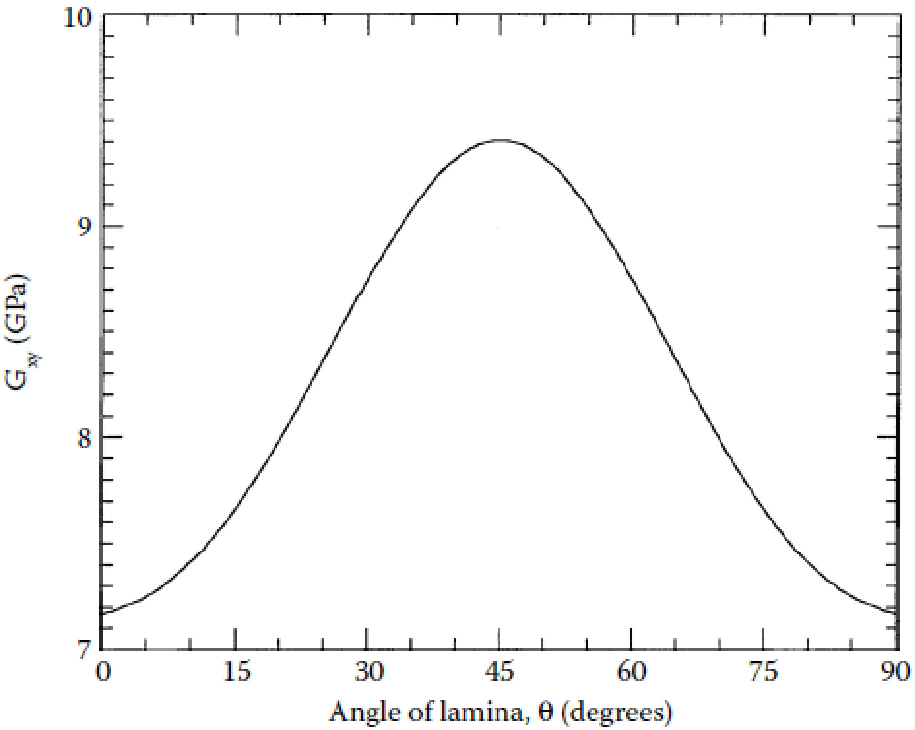
The variations of the Young's modulus, E_x and E_y are inverses of each other.

Examples

Poisson's ratio ν_{xy} as a function of angle of lamina for a graphite/epoxy lamina.

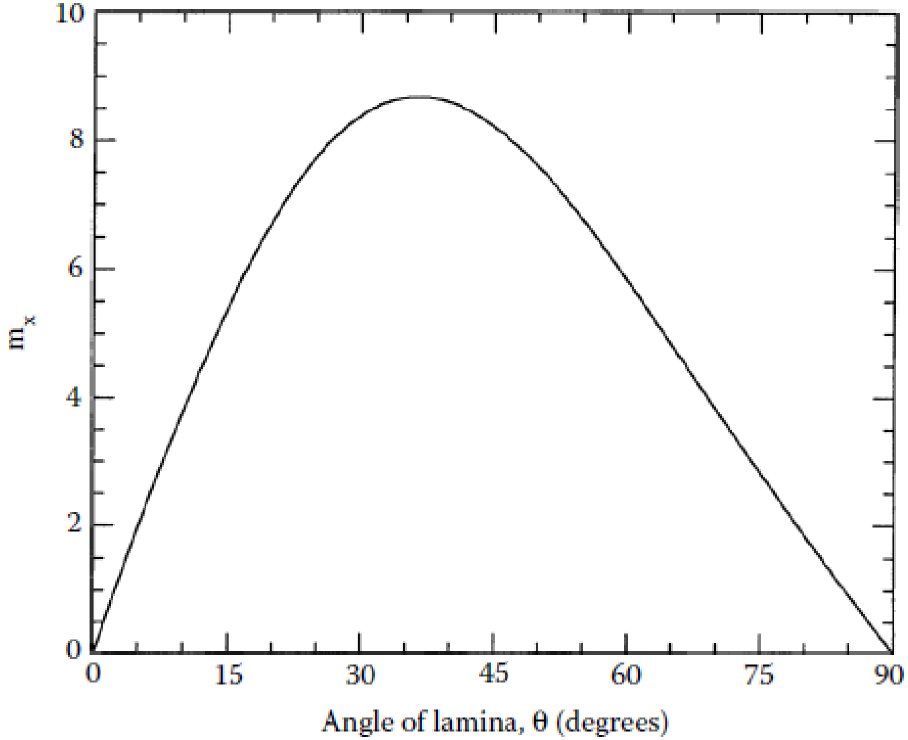
In-plane shear modulus in xy-plane as a function of angle of lamina for a graphite/epoxy lamina.



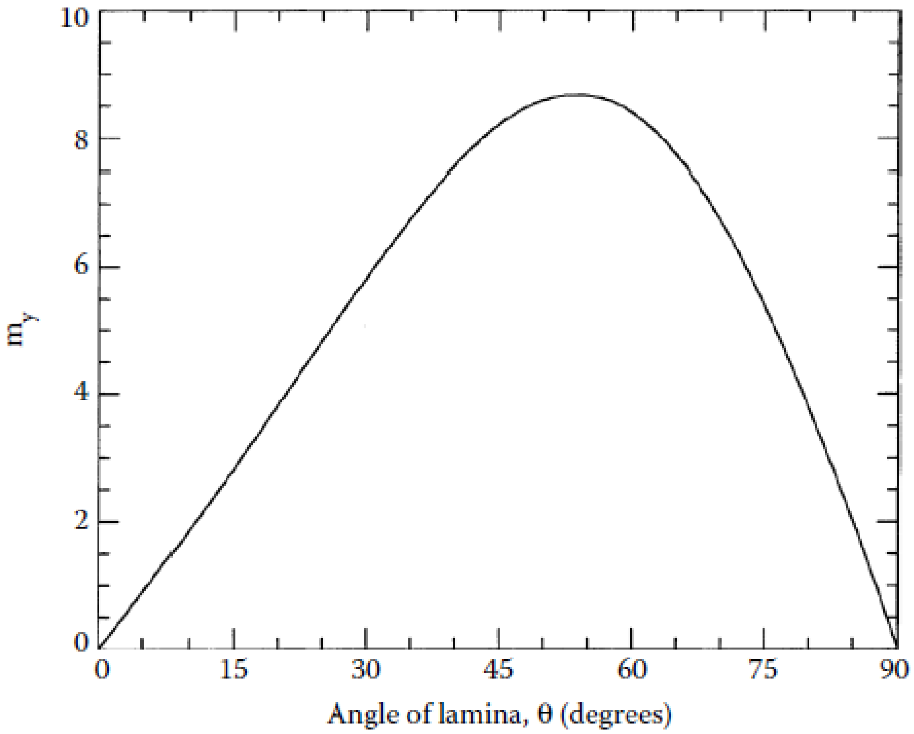


Examples

Shear coupling coefficient m_x as a function of angle of lamina for a graphite/epoxy lamina.



Shear coupling coefficient m_y as a function of angle of lamina for a graphite/epoxy lamina.



2.6. Strength Failure Theories of an Angle Lamina

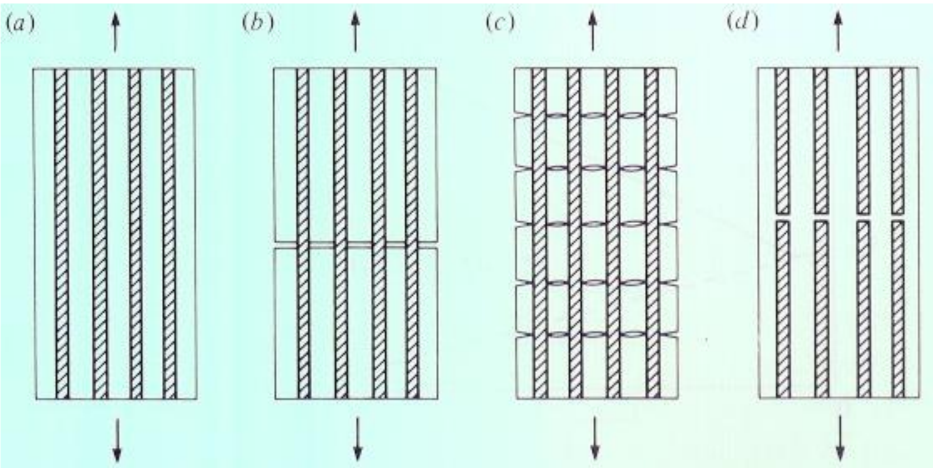
- A successful design of a structure requires efficient and safe use of materials. Theories need to be developed to compare the state of stress in a material to failure criteria. It should be noted that failure theories are only stated and their application is validated by experiments.
- For a laminate, the strength is related to the strength of each individual lamina. Various theories have been developed for studying the failure of an angle lamina. The theories are generally based on the normal and shear strengths of a unidirectional lamina.
- A simple failure theory for an isotropic material is based on finding the principal normal stresses and the maximum shear stresses. These maximum stresses, if greater than any of the corresponding ultimate strengths, indicate failure in the material.
- However, in a lamina, the failure theories are not based on principal normal stresses and maximum shear stresses. Rather, they are based on the stresses in the material or local axes because a lamina is orthotropic and its properties are different at different angles.

2.6. Strength Failure Theories of an Angle Lamina

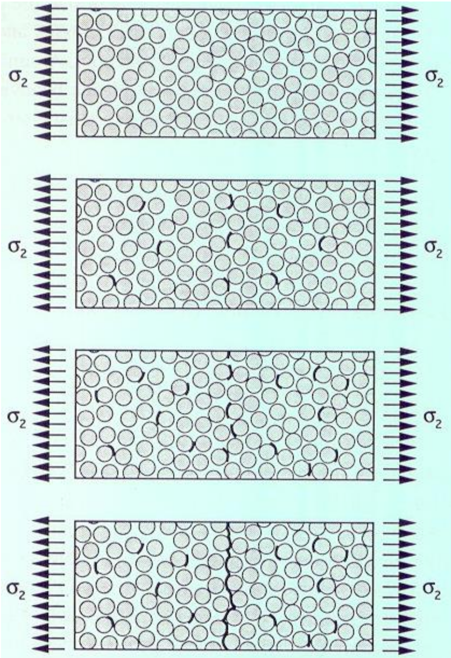
Failures of a unidirectional lamina

Longitudinal tensile failure

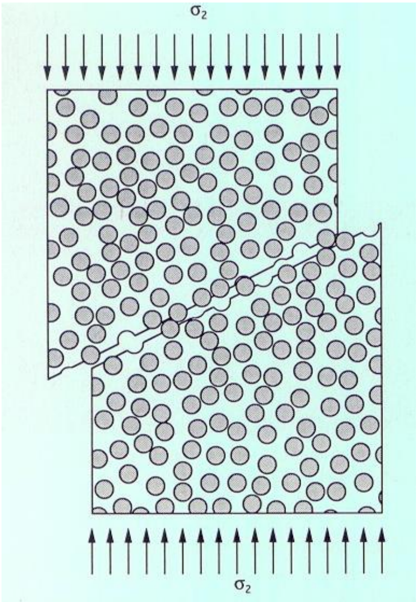
(a) before fracture, (b) resin fracture before fiber fracture, (c) multiple resin cracking, (d) fiber fracture before resin fracture.



Transverse tensile failure

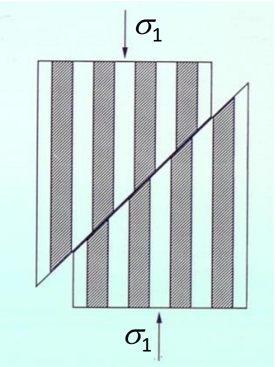


Transverse compression failure



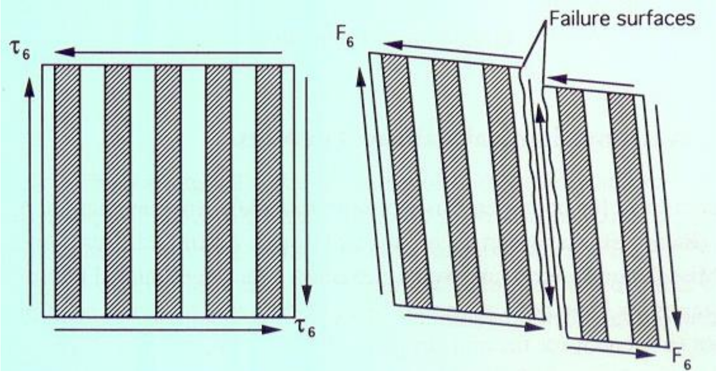
Longitudinal compression failure

Fiber buckling, Shear failure mode



Failure by in-plane shear

Due to stress concentration at fibre-matrix interface



✓ The strengths or ultimate stresses are denoted throughout this lecture as follows:

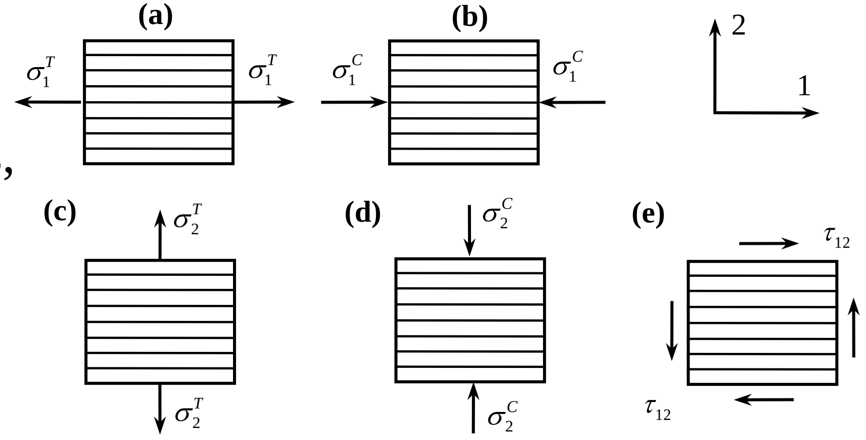
$\left(\sigma_1^T\right)_{ult}$: ultimate longitudinal tensile stress,

$\left(\sigma_1^C\right)_{ult}$: ultimate longitudinal compressive stress,

$\left(\sigma_2^T\right)_{ult}$: ultimate transverse tensile stress,

$\left(\sigma_2^C\right)_{ult}$: ultimate transverse compressive stress,

$\left(\tau_{12}\right)_{ult}$: ultimate in – plane shear stress.



✓ The ultimate strains are also denoted as follows:

$\left(\varepsilon_1^T\right)_{ult}$: ultimate longitudinal tensile strain,

$\left(\varepsilon_1^C\right)_{ult}$: ultimate longitudinal compressive strain,

$\left(\varepsilon_2^T\right)_{ult}$: ultimate transverse tensile strain,

$\left(\varepsilon_2^C\right)_{ult}$: ultimate transverse compressive strain,

$\left(\gamma_{12}\right)_{ult}$: ultimate in – plane shear strain.

- ✓ Failure theories for composite lamina can be classified into three distinct groups.
  - Non-interactive or limit theories: Here, failure modes are predicted by comparing individual stresses or strains respective to their ultimate stresses and strains. However, such theories do not take into account the interaction among different stress components. Examples of such theories are maximum stress criteria, and maximum strain criteria.
  - Interactive theories: These theories go a step further than limit theories, and account for interaction between various stress/strain components. Examples of such theories are those of Tsai-Wu and Tsai-Hill. They are able to predict overall failure, but cannot predict the exact failure mode.
  - Failure mode based theories: These theories provide separate criteria for failure of matrix, fiber and interface. Examples of such theories are those of Puck, and Hashim-Rotem.
- ✓ In this course, we are going to discuss four theories including maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu.

2.6.1 Maximum Stress Failure Theory

- ✓ This theory is similar to the maximum normal stress theory by Rankine and the maximum shear stress theory by Tresca.
- ✓ Failure is predicted in a lamina, if any of the normal or shear stresses in the local axes of a lamina is equal to or exceeds the corresponding ultimate strengths of the unidirectional lamina.
- ✓ This theory is written mathematically and stated as follows: A lamina is considered to be failed if

$$
\begin{aligned}
&-\left(\sigma_1^C\right)_{ult} < \sigma_1 < \left(\sigma_1^T\right)_{ult}, \text{ or}\\
&-\left(\sigma_2^C\right)_{ult} < \sigma_2 < \left(\sigma_2^T\right)_{ult}, \text{ or}\\
&-\left(\tau_{12}\right)_{ult} < \tau_{12} < \left(\tau_{12}\right)_{ult}
\end{aligned}
\qquad (2.6.1)
$$

is violated.

- ✓ Note that all five strength parameters are treated as positive numbers, and the normal stresses are positive if tensile and negative if compressive.

✓ The failure envelope for this theory under biaxial normal loading is clearly illustrated in Figure 2.6.1. The advantage of this theory is that it is simple to use but the major disadvantage is that there is no interaction between the stress components.

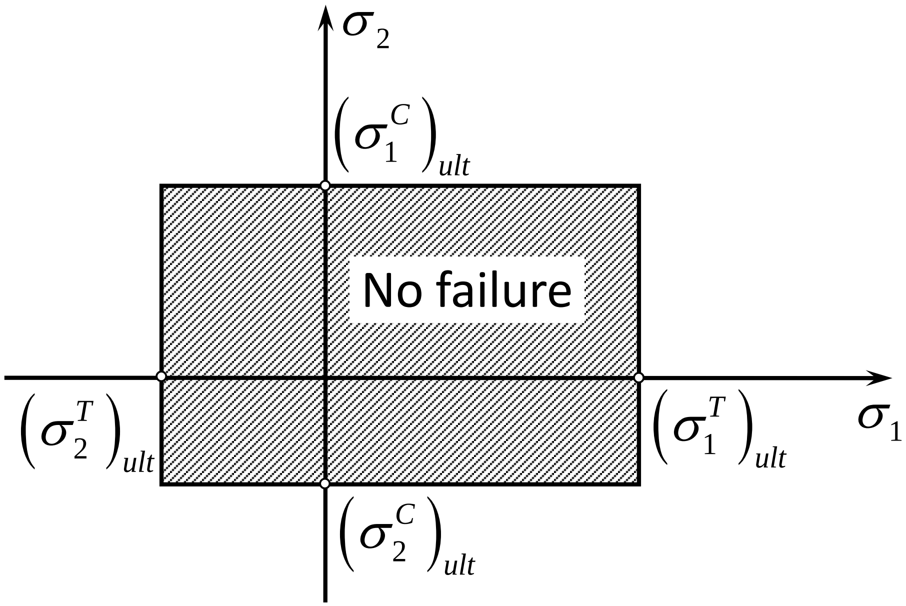


Figure 2.6.1. Failure envelope for the maximum stress failure theory under biaxial normal loading.

Strength Ratio

- In a failure theory such as the maximum stress failure theory of Section 2.6.1, it can be determined whether a lamina has failed if any of the inequalities of Equation (2.6.1) are violated. However, this does not give the information about how much the load can be increased if the lamina is safe or how much the load should be decreased if the lamina has failed. The definition of strength ratio (SR) is helpful here. The strength ratio is defined as

$$SR = \frac{\text{Maximum Load Which Can Be Applied}}{\text{Load Applied}}$$

- The concept of strength ratio is applicable to any failure theory. If SR>1, then the lamina is safe and the applied stress can be increased by a factor of SR. If SR<1, the lamina is unsafe and the applied stress needs to be reduced by a factor of SR. A value of SR = 1 implies the failure load.

2.6.2 Maximum Strain Failure Theory

- ✓ This theory is based on the maximum normal strain theory by St. Venant and the maximum shear strain theory by Tresca.
- ✓ The strains applied to a lamina are resolved to strains in the local axes.
- ✓ Failure is predicted in a lamina, if any of the normal or shear strains in the local axes of a lamina equal to or exceed the corresponding ultimate strains of the unidirectional lamina.
- ✓ A lamina is considered to be failed if

$$\begin{aligned}
&-\left(\varepsilon_1^C\right)_{ult} < \varepsilon_1 < \left(\varepsilon_1^T\right)_{ult}, \ \text{or} \\
&-\left(\varepsilon_2^C\right)_{ult} < \varepsilon_2 < \left(\varepsilon_2^T\right)_{ult}, \ \text{or} \qquad (2.6.2) \\
&-\left(\gamma_{12}\right)_{ult} < \gamma_{12} < \left(\gamma_{12}\right)_{ult}
\end{aligned}$$

is violated.

- ✓ The ultimate strains can be found directly from the ultimate strength parameters and the elastic moduli. Assuming the stress–strain response is linear until failure, we have the following relations between the strains and the stresses in the longitudinal direction as well as in the transverse direction:

$$\left(\sigma_1^T\right)_{ult} = \sigma_1 - \nu_{12}\sigma_2$$

$$\left(\sigma_2^T\right)_{ult} = \sigma_2 - \nu_{21}\sigma_1$$

- ✓ The advantage of this theory is simple to use but the major disadvantage is no interaction between the strain components.

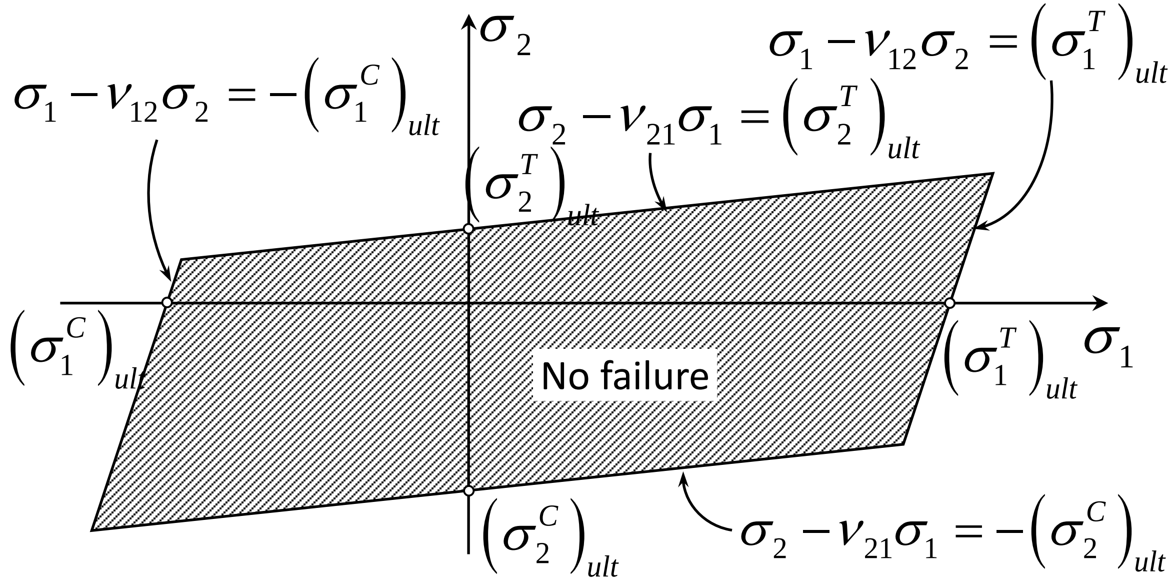


Figure 2.6.2. Failure envelope for the maximum strain failure theory under biaxial normal loading.

2.6.3 Tsai–Hill Failure Theory

- ✓ This theory is based on the distortion energy failure theory of Von-Mises' distortional energy yield criterion. For 2D state of stress, the von Mises yield criterion has the form

$$\sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2 = \sigma_{yp}$$

- ✓ Hill modified this criterion for the case of ductile metals with anisotropy

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 + D\tau_6^2 = 1$$

- ✓ Based on the distortion energy theory, Tsai and Hill proposed that a lamina has failed if

$$\begin{aligned}&(G_2 + G_3)\sigma_1^2 + (G_1 + G_3)\sigma_2^2 + (G_1 + G_2)\sigma_3^2 - 2G_3\sigma_1\sigma_2 \\ &-2G_2\sigma_1\sigma_3 - 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 \le 1\end{aligned} \qquad (2.6.3)$$

is violated.

- ✓ The components G_1, G_2, G_3, G_4, G_5, and G_6 of the strength criterion depend on the failure strengths

2.6. Strength Failure Theories of an Angle Lamina

- ✓ The components G_1, G_2, G_3, G_4, G_5, and G_6 are found as follows

1. Apply $\sigma_1 = \left(\sigma_1^T\right)_{ult}$ to a unidirectional lamina, then the lamina will fail. Hence, Equation (2.6.3) reduces to: $(G_2 + G_3)\left(\sigma_1^T\right)_{ult}^2 = 1$ (2.6.4)
2. Apply $\sigma_2 = \left(\sigma_2^T\right)_{ult}$: $(G_1 + G_3)\left(\sigma_2^T\right)_{ult}^2 = 1$ (2.6.5)
3. Apply $\sigma_3 = \left(\sigma_2^T\right)_{ult}$, and assuming that the normal tensile failure strength is the same in direction (2) and (3): $(G_1 + G_2)\left(\sigma_2^T\right)_{ult}^2 = 1$ (2.6.6)
4. Apply $\tau_{12} = \left(\tau_{12}\right)_{ult}$: $2G_6\left(\tau_{12}^S\right)_{ult}^2 = 1$ (2.6.7)

- ✓ From Equation (2.6.4) to Equation (2.6.7), we obtain

$$G_1 = \frac{1}{2}\left[\frac{2}{\left(\sigma_2^T\right)_{ult}^2} - \frac{1}{\left(\sigma_1^T\right)_{ult}^2}\right],\quad G_2 = \frac{1}{2}\left[\frac{1}{\left(\sigma_1^T\right)_{ult}^2}\right],\quad G_3 = \frac{1}{2}\left[\frac{1}{\left(\sigma_1^T\right)_{ult}^2}\right],\quad G_6 = \frac{1}{2}\left[\frac{1}{\left(\tau_{12}\right)_{ult}^2}\right]. \qquad (2.6.8)$$

- ✓ Because the unidirectional lamina is assumed to be under plane stress – that is, $\sigma_3 = \tau_{31} = \tau_{23} = 0$, then Equation (2.6.3) reduces to

$$\left[\frac{\sigma_1}{\left(\sigma_1^T\right)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{\left(\sigma_1^T\right)_{ult}^2}\right] + \left[\frac{\sigma_2}{\left(\sigma_2^T\right)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{\left(\tau_{12}\right)_{ult}}\right]^2 \le 1 \qquad (2.6.9)$$

- ✓ The advantage of this theory is that there is interaction between the stress components. However, this theory does not distinguish between the tensile and compressive strengths and is not as simple to use as the maximum stress theory or the maximum strain theory.

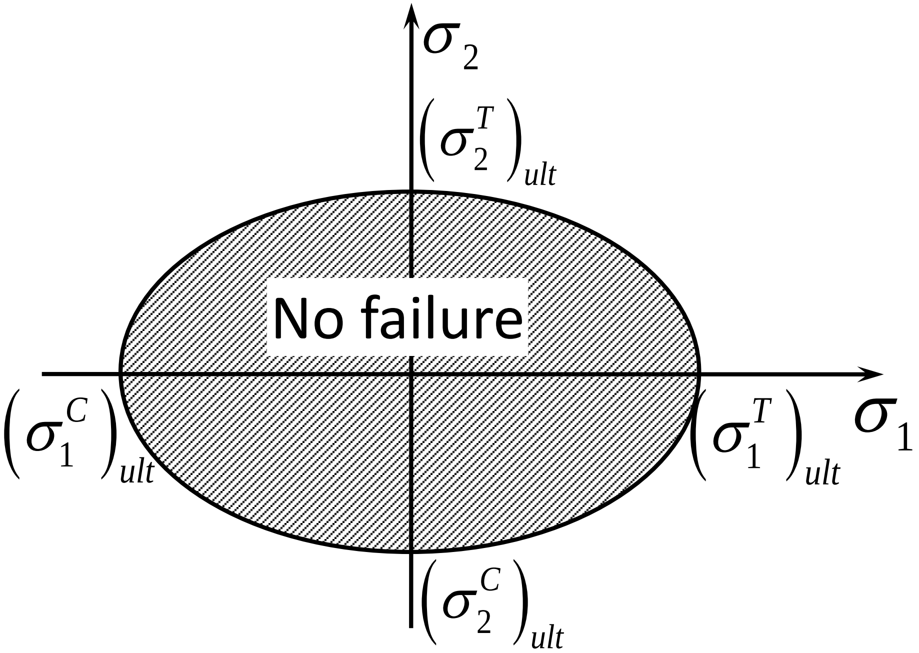


Figure 2.6.3. Failure envelope for the Tsai-Hill failure theory.

$$\left[\frac{\sigma_1}{\left(\sigma_1^T\right)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{\left(\sigma_1^T\right)_{ult}^2}\right] + \left[\frac{\sigma_2}{\left(\sigma_2^T\right)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{\left(\tau_{12}\right)_{ult}}\right]^2 \le 1$$

Modified Tsai–Hill failure theory

- ✓ Tsai–Hill failure theory underestimates the failure stress because the transverse tensile strength of a unidirectional lamina is generally much less than its transverse compressive strength.
- ✓ The compressive strengths are not used in the Tsai–Hill failure theory, but it can be modified to use corresponding tensile or compressive strengths in the failure theory as follows

$$\left[\frac{\sigma_1}{X_1}\right]^2 - \left[\left(\frac{\sigma_1}{X_2}\right)\left(\frac{\sigma_2}{X_2}\right)\right] + \left[\frac{\sigma_2}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 \leq 1 \qquad (2.6.10)$$

where

$$X_1 = \left(\sigma_1^T\right)_{ult}, if\ \sigma_1 > 0 \qquad X_2 = \left(\sigma_1^T\right)_{ult}, if\ \sigma_2 > 0 \qquad Y = \left(\sigma_2^T\right)_{ult}, if\ \sigma_2 > 0$$

$$= \left(\sigma_1^C\right)_{ult}, if\ \sigma_1 < 0 \qquad = \left(\sigma_1^C\right)_{ult}, if\ \sigma_2 < 0 \qquad = \left(\sigma_2^C\right)_{ult}, if\ \sigma_2 < 0$$

$$S = \left(\tau_{12}\right)_{ult}$$

Advantages and Drawbacks of Tsai–Hill Failure Theory

- Unlike the Maximum Strain and Maximum Stress Failure Theories, the Tsai-Hill failure theory considers the interaction among the three unidirectional lamina strength parameters.

- The Tsai-Hill Failure Theory does not distinguish between the compressive and tensile strengths in its equation. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories.

- The Tsai–Hill failure theory is a unified theory and thus does not give the mode of failure like the maximum stress and maximum strain failure theories do. However, one can make a reasonable guess of the failure mode by calculating $\left|\sigma_1 / \left(\sigma_1^T\right)_{ult}\right|, \left|\sigma_2 / \left(\sigma_2^T\right)_{ult}\right|$, and $\left|\tau_{12} / \left(\tau_{12}\right)_{ult}\right|$

2.6.4 Tsai-Wu Failure Theory

- ✓ The Tsai-Wu failure theory is based on polynomial function:

$$H_i \sigma_i + H_{ij} \sigma_i \sigma_j = 1$$

where H_i and H_{ij} are strength tensors of second-, and fourth- order strength tensor, and $i, j = 1, 2, \ldots, 6$.

- ✓ Tsai-Wu applied the failure theory to a lamina in plane stress. In this theory, failure is assumed to occur in the lamina if the following condition is satisfied:

$$H_1 \sigma_1 + H_2 \sigma_2 + H_6 \tau_{12} + H_{11} \sigma_1^2 + H_{22} \sigma_2^2 + H_{66} \tau_{12}^2 + 2H_{12} \sigma_1 \sigma_2 \leq 1 \quad (2.6.11)$$

- ✓ This failure theory is more general than the Tsai–Hill failure theory because it distinguishes between the compressive and tensile strengths of a lamina.

- ✓ The components H_1, H_2, H_6, H_{11}, H_{22}, and H_{66} of the failure theory are found using the five strength parameters of a unidirectional lamina.

2.6. Strength Failure Theories of an Angle Lamina

✓ The components H_1, H_2, H_6, H_{11}, H_{22}, and H_{66} are found as follows

1. Apply $\sigma_1 = \left(\sigma_1^T\right)_{ult}$ to a unidirectional lamina, then the lamina will fail. Hence, Equation (2.6.11) reduces to:

$$H_1\left(\sigma_1^T\right)_{ult} + H_{11}\left(\sigma_1^T\right)_{ult}^2 = 1 \quad (2.6.12) \qquad H_1 = \frac{1}{\left(\sigma_1^T\right)_{ult}} - \frac{1}{\left(\sigma_1^C\right)_{ult}},$$

2. Apply $\sigma_1 = -\left(\sigma_1^C\right)_{ult}$: $-H_1\left(\sigma_1^C\right)_{ult} + H_{11}\left(\sigma_1^C\right)_{ult}^2 = 1$ (2.6.13) $\qquad H_{11} = \frac{1}{\left(\sigma_1^T\right)_{ult}\left(\sigma_1^C\right)_{ult}},$

3. Apply $\sigma_2 = \left(\sigma_2^T\right)_{ult}$: $\quad H_2\left(\sigma_2^T\right)_{ult} + H_{22}\left(\sigma_2^T\right)_{ult}^2 = 1$ (2.6.14) $\qquad H_2 = \frac{1}{\left(\sigma_2^T\right)_{ult}} - \frac{1}{\left(\sigma_2^C\right)_{ult}},$

4. Apply $\sigma_2 = -\left(\sigma_2^C\right)_{ult}$: $-H_2\left(\sigma_2^C\right)_{ult} + H_{22}\left(\sigma_2^C\right)_{ult}^2 = 1$ (2.6.15) $\qquad H_{22} = \frac{1}{\left(\sigma_2^T\right)_{ult}\left(\sigma_2^C\right)_{ult}}$

5. Apply $\tau_{12} = \left(\tau_{12}\right)_{ult}$: $\quad H_6\left(\tau_{12}\right)_{ult} + H_{66}\left(\tau_{12}^S\right)_{ult}^2 = 1$ (2.6.16) $\qquad H_6 = 0,$

6. Apply $\tau_{12} = -\left(\tau_{12}\right)_{ult}$: $-H_6\left(\tau_{12}\right)_{ult} + H_{66}\left(\tau_{12}^S\right)_{ult}^2 = 1$ (2.6.17) $\qquad H_{66} = \frac{1}{\left(\tau_{12}\right)_{ult}^2}.$

- ✓ The components H_{12} is found as follows

The component H_{12} can be found experimentally by knowing a biaxial stress at which the lamina fails. Experimental methods to find H_{12} :

1. Apply equal tensile loads along the two material axes in a unidirectional composite. If $\sigma_x = \sigma_y = \sigma$, $\tau_{xy} = 0$ is the load at which the lamina fails, then

$$\left(H_1 + H_2\right)\sigma + \left(H_{11} + H_{22} + 2H_{12}\right)\sigma^2 = 1. \qquad (2.6.18)$$

The solution of the Equation (2.6.18) gives:

$$\boxed{H_{12} = \frac{1}{2\sigma^2}\left[1\text{-}(H_1 + H_2)\sigma - (H_{11} + H_{22})\sigma^2\right].} \qquad (2.6.19)$$

It is not necessary to pick tensile loads in the preceding biaxial test, but one may apply any combination of

$$\sigma_1 = \sigma,\ \sigma_2 = \sigma, \qquad \sigma_1 = -\sigma,\ \sigma_2 = -\sigma,$$

$$\sigma_1 = \sigma,\ \sigma_2 = -\sigma, \qquad \sigma_1 = -\sigma,\ \sigma_2 = \sigma,$$

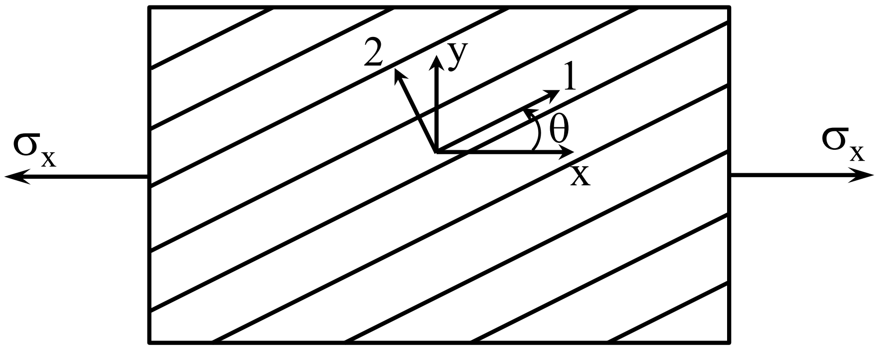
This will give four different values of H_{12}

Moreover, direct biaxial testing is not easy or practical to perform.

2.6. Strength Failure Theories of an Angle Lamina

✓ The components H_{12} is found as follows

2. An easier test producing a biaxial state of stress is the off-axis tensile test, that is, uniaxial loading or at an angle θ to the fiber direction on the 1-2 plane. For example θ = 45°, if uniaxial stress is $\sigma_x = \sigma$, then,



$$\sigma_1 = \frac{\sigma}{2}, \qquad \sigma_2 = \frac{\sigma}{2}, \qquad \tau_{12} = -\frac{\sigma}{2}. \tag{2.6.20}$$

Substituting the preceding local stresses in Equation (2.6.11),

$$\left(H_1 + H_2\right)\frac{\sigma}{2} + \frac{\sigma^2}{4}\left(H_{11} + H_{22} + H_{66} + 2H_{12}\right) = 1, \tag{2.6.21}$$

$$\boxed{H_{12} = \frac{2}{\sigma^2} - \frac{\left(H_1 + H_2\right)}{\sigma} - \frac{1}{2}\left(H_{11} + H_{22} + H_{66}\right).} \tag{2.6.22}$$

2.6. Strength Failure Theories of an Angle Lamina

Some empirical suggestions for finding the value of *H_{12}*

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2}$$ as per Tsai-Hill failure theory

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$ as per Hoffman criterion (2.6.23a-c)

$$H_{12} = -\frac{1}{2}\sqrt{\frac{1}{(\sigma_1^T)_{ult}\,(\sigma_1^C)_{ult}\,(\sigma_2^T)_{ult}\,(\sigma_2^C)_{ult}}}$$ as per Mises-Hencky criterion

✓ The advantage of this theory is that there is interaction between the stress components and the theory does distinguish between the tensile and compressive strengths. A major disadvantage of this theory is that it is not simple to use.

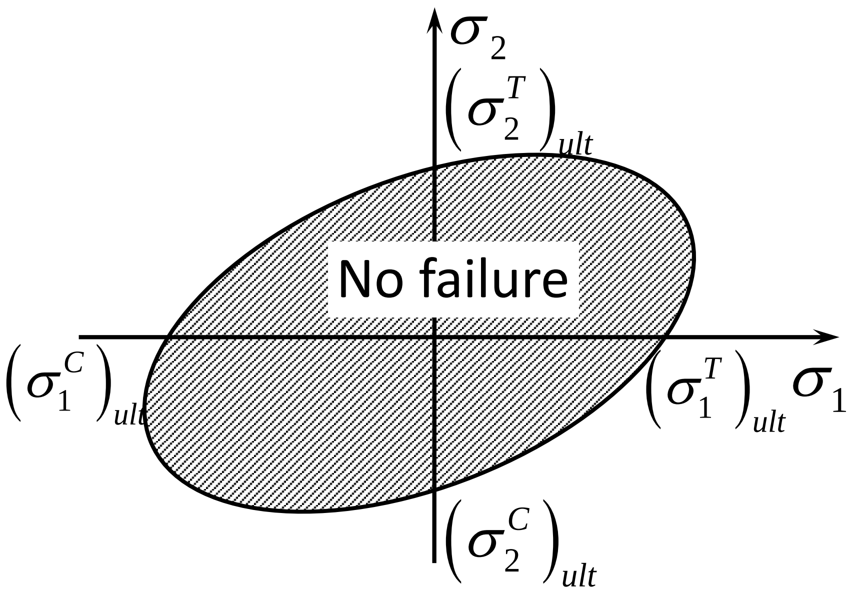


Figure 2.6.4. Failure envelope for the Tsai-Wu failure theory.

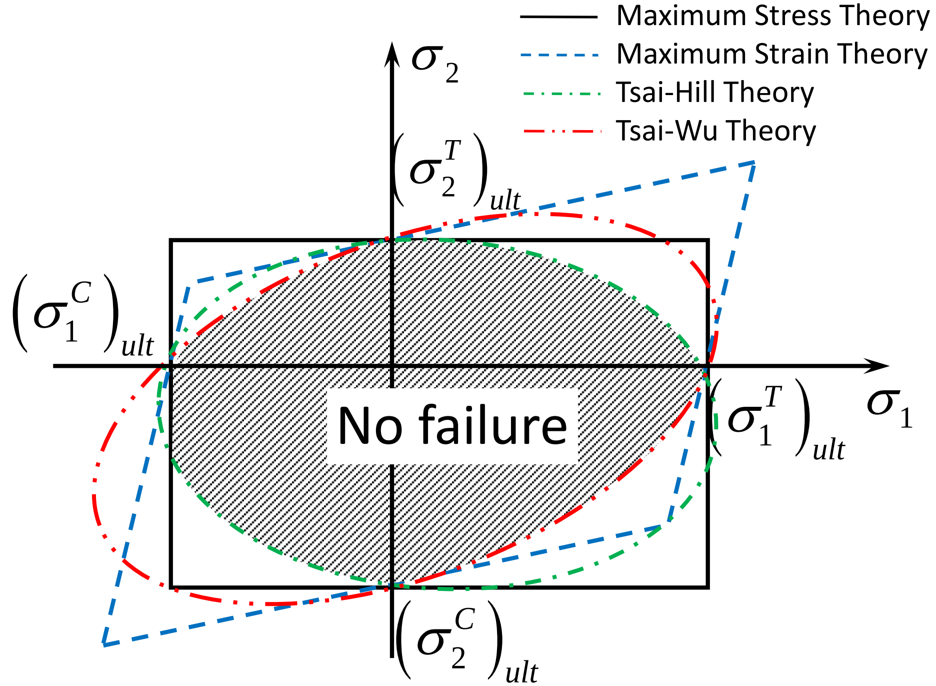


Figure 2.6.5. Comparison between the four failure envelopes.

2.7. Comparison of Failure Theories

- Tsai compared the results from various failure theories to some experimental results. He considered an angle lamina subjected to a uniaxial load in the x-direction, σ_x, as shown in Figure 2.7.1.
- The failure stresses were obtained experimentally for tensile and compressive stresses for various angles of the lamina.

$$\sigma_1 = \sigma_x \cos^2\theta,$$

$$\sigma_2 = \sigma_x \sin^2\theta,$$

$$\tau_{12} = -\sigma_x \cos\theta \sin\theta$$

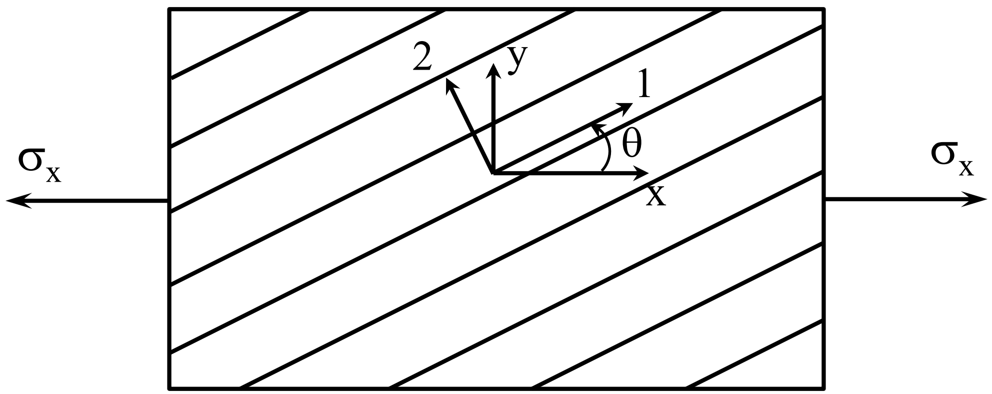


Figure 2.7.1. Off-axis loading in the x-direction.

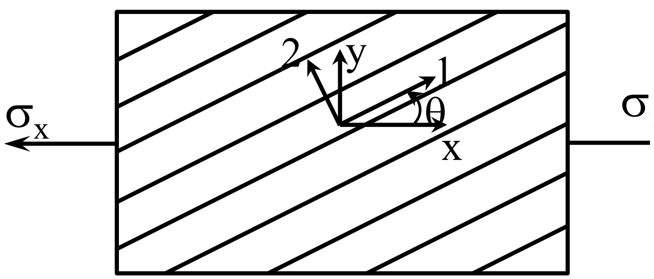
- Using the preceding local strains and stresses in the four failure theories given by Eq. (2.6.1), Eq. (2.6.2), Eq. (2.6.9), and Eq. (2.6.11), one can find the ultimate off-axis load, σ_x, that can be applied as a function of the lamina angle.

Experimental Results and Maximum Stress Failure Theory

$$-\left(\sigma_1^C\right)_{ult} < \sigma_1 < \left(\sigma_1^T\right)_{ult}, \text{ or}$$

$$-\left(\sigma_2^C\right)_{ult} < \sigma_2 < \left(\sigma_2^T\right)_{ult}, \text{ or}$$

$$-\left(\tau_{12}\right)_{ult} < \tau_{12} < \left(\tau_{12}\right)_{ult}$$



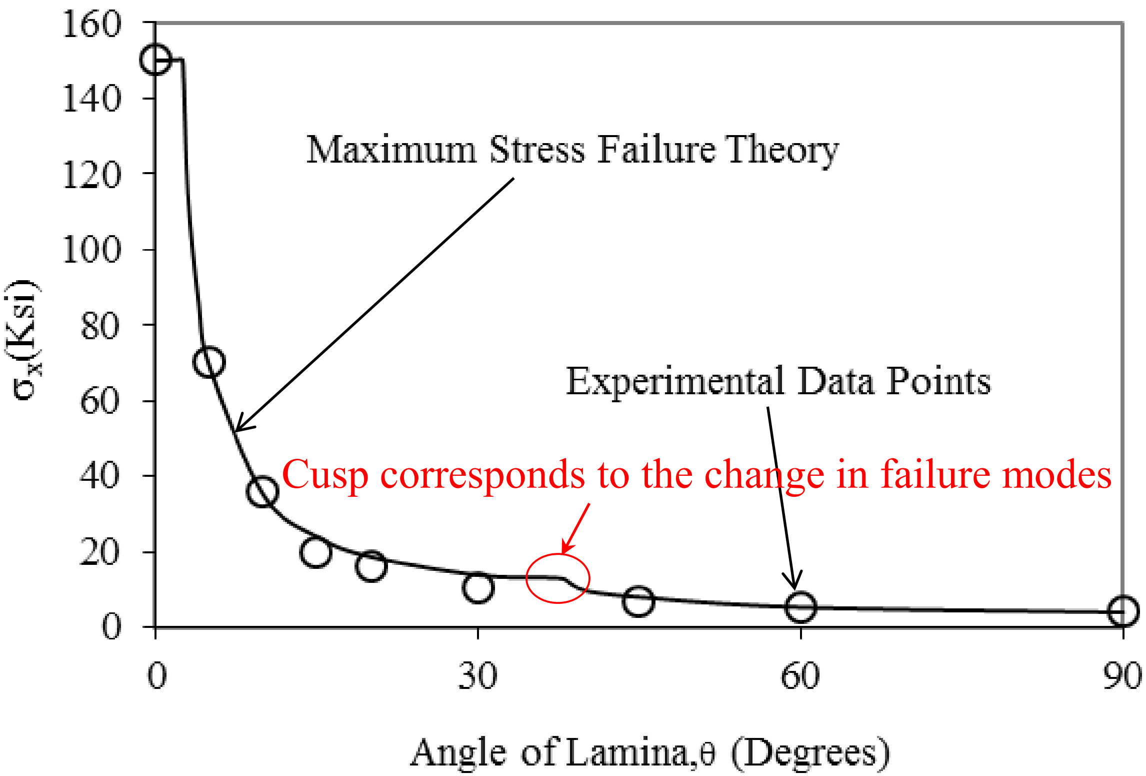


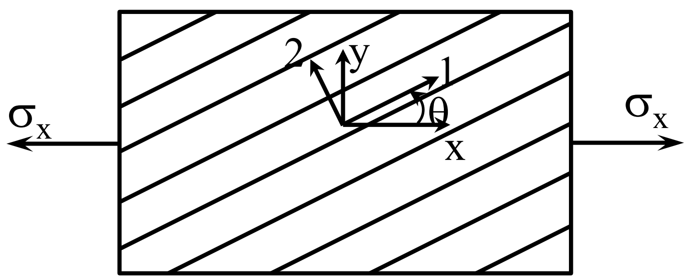
Figure 2.7.2. Maximum normal tensile stress in the x-direction as a function of angle of lamina using maximum stress failure theory. (Experimental data reprinted with permission from Tsai et al. [21])

Experimental Results and Maximum Strain Failure Theory

$$-\left(\varepsilon_1^C\right)_{ult} < \varepsilon_1 < \left(\varepsilon_1^T\right)_{ult}, \text{ or}$$

$$-\left(\varepsilon_2^C\right)_{ult} < \varepsilon_2 < \left(\varepsilon_2^T\right)_{ult}, \text{ or}$$

$$-\left(\gamma_{12}\right)_{ult} < \gamma_{12} < \left(\gamma_{12}\right)_{ult}$$



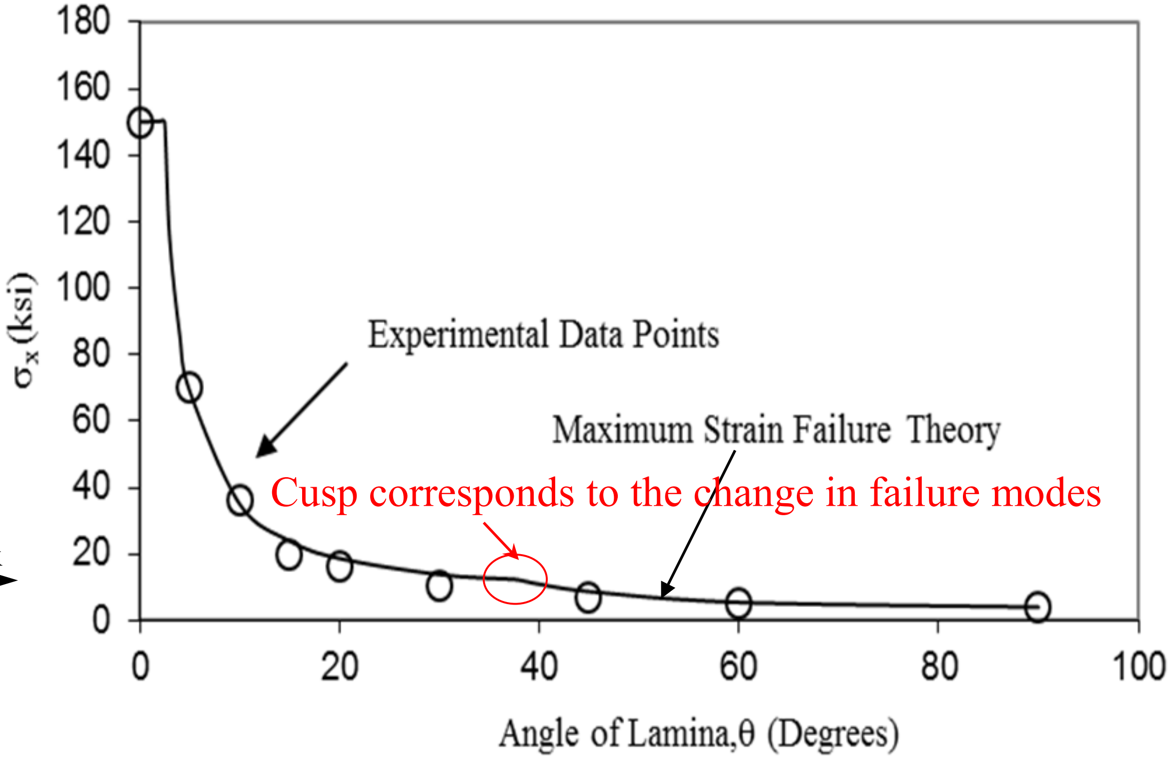
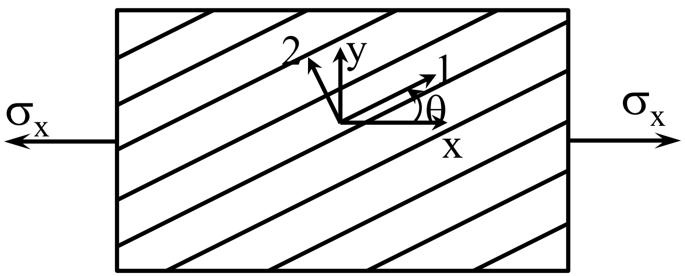


Figure 2.7.3. Maximum normal tensile stress in the x-direction as a function of angle of lamina using maximum strain failure theory. (Experimental data reprinted with permission from Tsai et al. [21])

Experimental Results and Tsai-Hill Failure Theory

$$\left[\frac{\sigma_1}{\left(\sigma_1^T\right)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{\left(\sigma_1^T\right)_{ult}^2}\right] + \left[\frac{\sigma_2}{\left(\sigma_2^T\right)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{\left(\tau_{12}\right)_{ult}}\right]^2 \leq 1$$



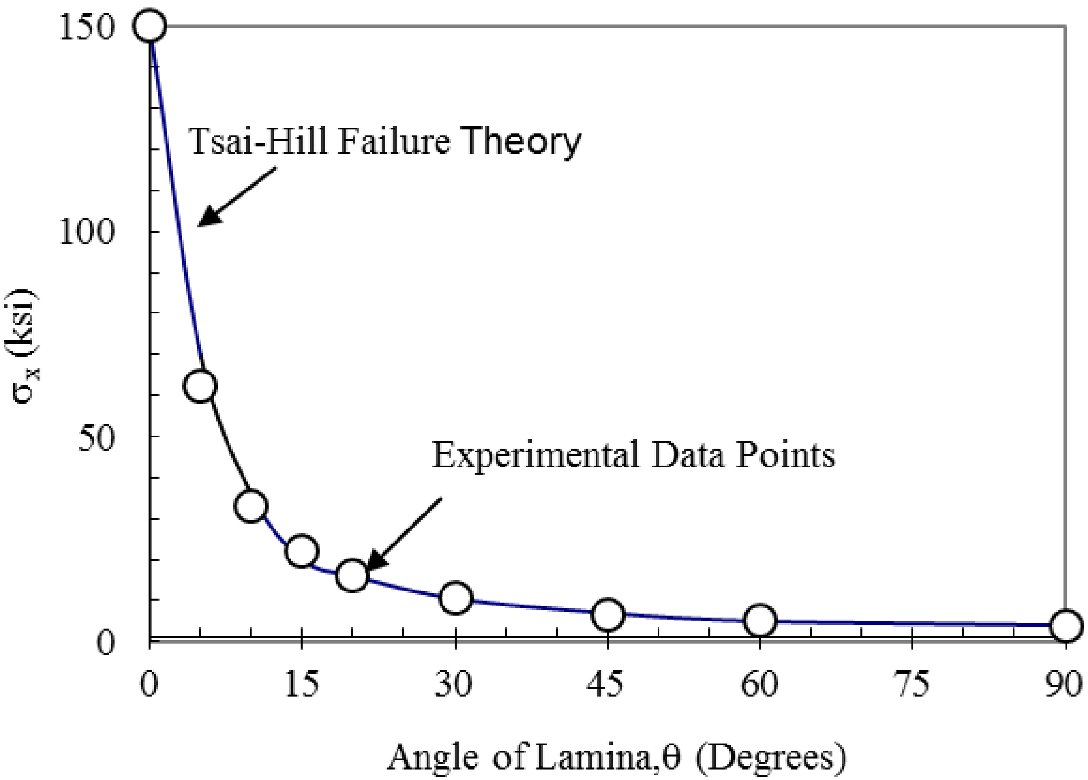
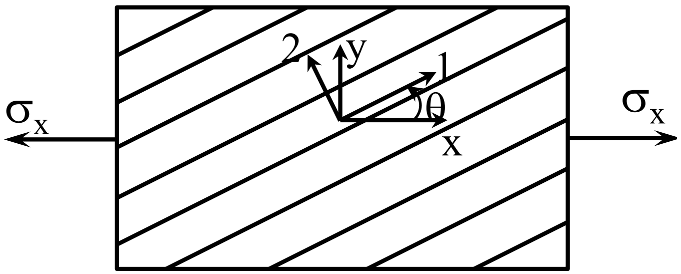


Figure 2.7.4. Maximum normal tensile stress in the x-direction as a function of angle of lamina using Tsai–Hill failure theory.

Experimental Results and Tsai-Wu Failure Theory

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\sigma_1^2 + H_{22}\sigma_2^2 + H_{66}\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 \leq 1$$



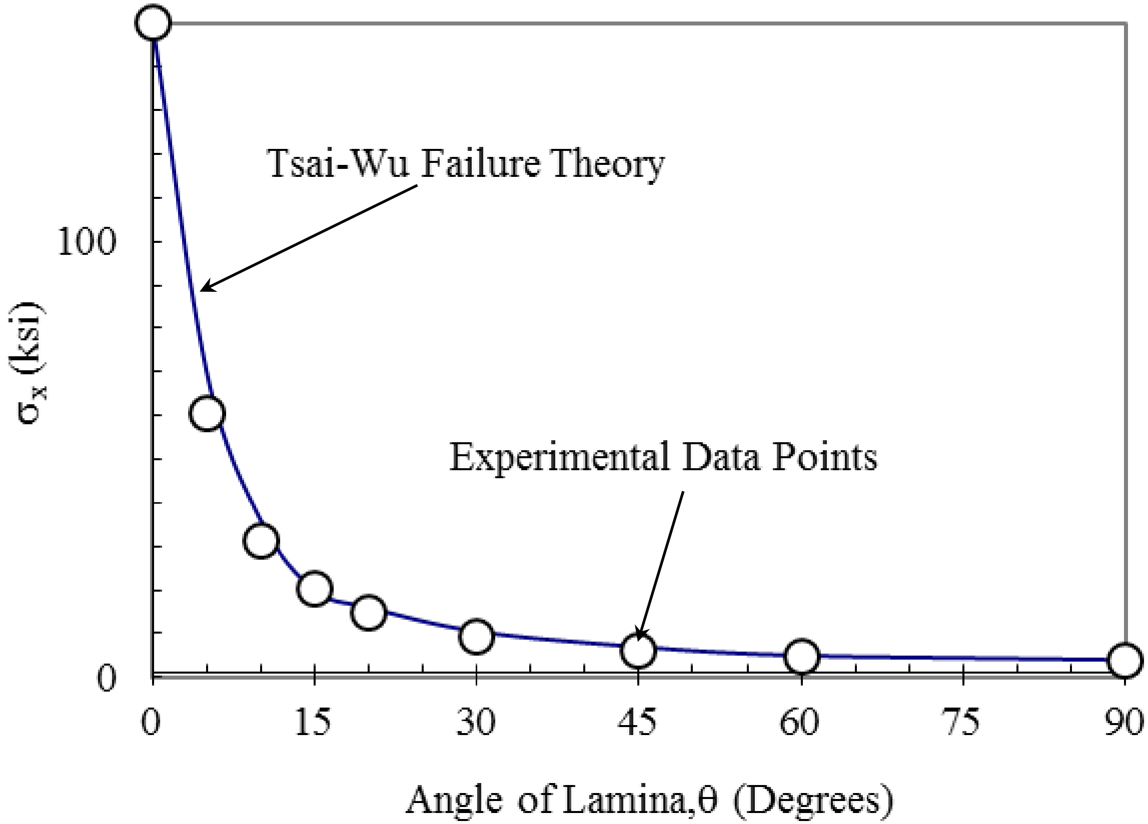


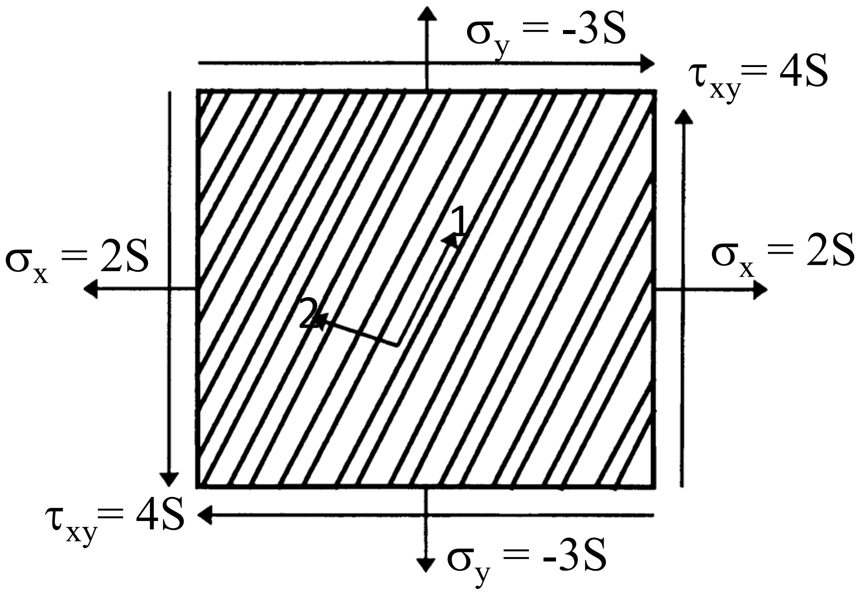
Figure 2.7.5. Maximum normal tensile stress in the x-direction as a function of angle of lamina using Tsai–Wu failure theory.

Example

Using four failure theories (maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu) to find the maximum value of S > 0 if a stress σ_x = 2S, σ_y = –3S, and τ_{xy} = 4S are applied to a 60° lamina of graphite/epoxy. The properties of a unidirectional graphite/epoxy lamina are given below.

$$\left(\sigma_1^T\right)_{ult} = 1500 \text{ MPa}$$
$$\left(\sigma_1^C\right)_{ult} = 1500 \text{ MPa}$$
$$\left(\sigma_2^T\right)_{ult} = 40 \text{ MPa}$$
$$\left(\sigma_2^C\right)_{ult} = 246 \text{ MPa}$$
$$\left(\tau_{12}\right)_{ult} = 68 \text{ MPa}$$

E_1 = 181 GPa,
E_2 = 10.3 GPa,
ν_{12} = 0.28,
G_{12} = 7.17 GPa.



Application of Failure Theory

1. First step is to calculate the stresses/strains in the material principal directions. This can be done by transformation of stresses from the global coordinates to local material coordinates of the ply.
2. Second step is to apply the failure criteria in the material coordinate system.

Example

1. Maximum Stress Failure Theory

The global and local stresses in an angle lamina are related to each other through the angle of the lamina, θ = 60°: $c = \cos\theta$, $s = \sin\theta$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.250 & 0.750 & 0.866 \\ 0.750 & 0.250 & -0.866 \\ -0.433 & 0.433 & 0.500 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$\sigma_1 = 1.714S, \; \sigma_2 = -2.714S, \; \tau_{12} = -4.165S$$

Using the inequalities (2.6.1) of maximum stress failure theory, we obtain

$$\begin{aligned} &-1500(10^6) < 1.714S < 1500(10^6), & & -875.1(10^6) < S < 875.1(10^6), \\ &-246(10^6) < -2.714S < 40(10^6), & \text{or} \quad & -14.73(10^6) < S < 90.64(10^6), \\ &-68(10^6) < -4.165S < 68(10^6). & & -16.33(10^6) < S < 16.33(10^6). \end{aligned}$$

All the inequality conditions (and S > 0) are satisfied if $0 < S < 16.33\,\text{MPa}.$

The above inequalities also show that the angle lamina will fail in shear. The maximum stress that can be applied before failure is:

$$\sigma_x = 32.66\,MPa, \; \sigma_y = -48.99\,MPa, \; \tau_{xy} = 65.32\,MPa.$$

Example

2. Maximum Strain Failure Theory

The local stresses in an angle lamina are:

$$\sigma_1 = 1.714S,\ \sigma_2 = -2.714S,\ \tau_{12} = -4.165S$$

The strains in the local axes are:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [S] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \qquad [S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \qquad \begin{cases} S_{11} = \dfrac{1}{E_1};\ S_{12} = -\dfrac{\nu_{12}}{E_1} \\ S_{22} = \dfrac{1}{E_2};\ S_{12} = -\dfrac{\nu_{21}}{E_2};\ S_{66} = \dfrac{1}{G_{12}} \end{cases}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.5525\times10^{-11} & 0.1547\times10^{-11} & 0 \\ 0.1547\times10^{-11} & 0.9709\times10^{-10} & 0 \\ 0 & 0 & 0.1395\times10^{-9} \end{bmatrix} \begin{bmatrix} 1.714 \\ -2.714 \\ -4.165 \end{bmatrix} S = \begin{bmatrix} 0.1367\times10^{-10} \\ -0.2662\times10^{-9} \\ -0.5809\times10^{-9} \end{bmatrix} S.$$

The ultimate failure strains are: $\left(\varepsilon_1^T\right)_{ult} = \dfrac{\left(\sigma_1^T\right)_{ult}}{E_1} = \dfrac{1500\times10^6}{181\times10^9} = 8.287\times10^{-3},$

$$\left(\varepsilon_2^T\right)_{ult} = \frac{\left(\sigma_2^T\right)_{ult}}{E_2} = \frac{40\times10^6}{10.3\times10^9} = 3.883\times10^{-3},\ \left(\varepsilon_1^C\right)_{ult} = \frac{\left(\sigma_1^C\right)_{ult}}{E_1} = \frac{1500\times10^6}{181\times10^9} = 8.287\times10^{-3},$$

$$\left(\varepsilon_2^C\right)_{ult} = \frac{\left(\sigma_2^C\right)_{ult}}{E_2} = \frac{246\times10^6}{10.3\times10^9} = 2.388\times10^{-2},\quad \left(\gamma_{12}\right)_{ult} = \frac{\left(\tau_{12}\right)_{ult}}{G_{12}} = \frac{68\times10^6}{7.17\times10^9} = 9.483\times10^{-3}.$$

Example

Using the inequalities (2.6.2) of the maximum strain failure theory, we obtain

$$-8.287\times10^{-3} < 0.1367\times10^{-10}S < 8.287\times10^{-3}, \qquad -606.2\times10^{6} < S < 606.2\times10^{6},$$

$$-2.388\times10^{-2} < -0.2662\times10^{-9}S < 3.883\times10^{-3}, \quad \text{Or} \quad -14.58\times10^{6} < S < 89.71\times10^{6},$$

$$-9.483\times10^{-3} < -0.5809\times10^{-9}S < 9.483\times10^{-3}, \qquad -16.33\times10^{6} < S < 16.33\times10^{6},$$

which give: $0 < S < 16.33\,\mathrm{MPa}$

- ✓ The maximum value of S before failure is 16.33 MPa. The same maximum value of S = 16.33 MPa is also found using maximum stress failure theory. There is no difference between the two values because the ***mode of failure is shear***.
- ✓ However, if the mode of failure were other than shear, a difference in the prediction of failure loads would have been present due to the Poisson's ratio effect, which couples the normal strains and stresses in the local axes.

Example

3. Tsai-Hill failure theory

The local stresses in an angle lamina are:

$$\sigma_1 = 1.714S,\ \sigma_2 = -2.714S,\ \tau_{12} = -4.165S$$

Using the Tsai-Hill failure theory from Equation (2.6.9), we obtain:

$$\left[\frac{1.714S}{1500\times10^6}\right]^2 - \left[\frac{1.714S\times(-2.714S)}{(1500\times10^6)^2}\right] + \left[\frac{-2.714S}{\underset{\uparrow\ (\sigma_2^T)_{ult}}{40\times10^6}}\right]^2 + \left[\frac{-4.165S}{68\times10^6}\right]^2 < 1$$

which give: $S < 10.94\,\mathrm{MPa}$

Using modified Tsai–Hill failure theory given by Eq. (2.6.10), we obtain:

$$\left[\frac{1.714S}{1500\times10^6}\right]^2 - \left[\frac{1.714S}{1500\times10^6}\right]\left[\frac{-2.714S}{1500\times10^6}\right] + \left[\frac{-2.714S}{\underset{\uparrow\ (\sigma_2^C)_{ult}}{246\times10^6}}\right]^2 + \left[\frac{-4.165S}{68\times10^6}\right]^2 < 1$$

which give: $S < 16.06\,\mathrm{MPa}$

The Tsai–Hill failure theory is a unified theory and does not give the mode of failure like the maximum stress and maximum strain failure theories do.

4. Tsai-Wu failure theory

The local stresses in an angle lamina are:

$$\sigma_1 = 1.714S,\ \sigma_2 = -2.714S,\ \tau_{12} = -4.165S$$

The components H_1, H_2, H_6, H_{11}, H_{22}, and H_{66} are found as follows:

$$H_1 = \frac{1}{1500\times10^6} - \frac{1}{1500\times10^6} = 0\ Pa^{-1}, \qquad H_2 = \frac{1}{40\times10^6} - \frac{1}{246\times10^6} = 2.093\times10^{-8}\ Pa^{-1},$$

$$H_6 = 0\ Pa^{-1}, \qquad H_{11} = \frac{1}{(1500\times10^6)(1500\times10^6)} = 4.4444\times10^{-19} Pa^{-2},$$

$$H_{22} = \frac{1}{(40\times10^6)(246\times10^6)} = 1.0162\times10^{-16} Pa^{-2}, \quad H_{66} = \frac{1}{(68\times10^6)^2} = 2.1626\times10^{-16} Pa^{-2},$$

Using the Mises–Hencky criterion for evaluation of H_{12}, (Eq. 2.6.23c):

$$H_{12} = -0.5\left[\left(4.4444\times10^{-19}\right)\left(1.0162\times10^{-16}\right)\right]^{\frac{1}{2}} = -3.360\times10^{-18} Pa^{-2}.$$

Example

Substituting these values in inequality (2.6.11), we obtain:

$$(0)(1.714S)+\left(2.093\times10^{-8}\right)(\text{-}2.714S)+(0)(\text{-}4.165S)$$
$$+\left(4.4444\times10^{-19}\right)(1.714S)^2+\left(1.0162\times10^{-16}\right)(\text{-}2.714S)^2$$
$$+\left(2.1626\times10^{-16}\right)(4.165S)^2+2\left(\text{-}3.360\times10^{-18}\right)(1.714S)(\text{-}2.714S)<1,$$

which give: $S < 22.39\,\mathrm{MPa}$

Using other two empirical criteria for H_{12} as per Equation (2.6.23), we obtain:

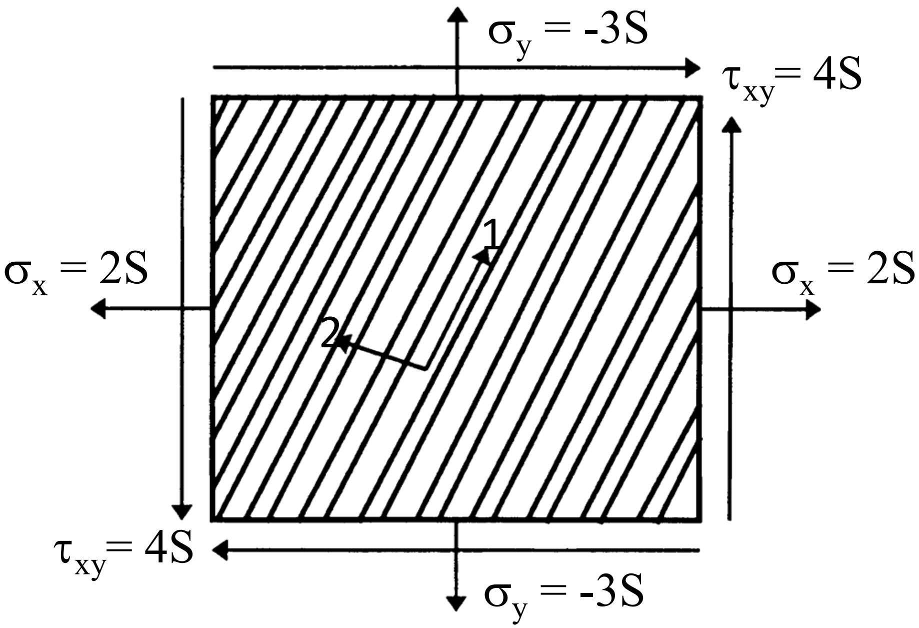
$$S < 22.49MPa \quad for \; H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2}$$ as per Tsai-Hill failure theory

$$S < 22.49MPa \quad for \; H_{12} = -\frac{1}{2\left(\sigma_1^T\right)_{ult}\left(\sigma_1^C\right)_{ult}}$$ as per Hoffman criterion

Example

The value of S for four failure theories for the same stress-state is:

$$
\begin{aligned}
S &= 16.33 \text{ (Maximum Stress failure theory)},\\
&= 16.33 \text{ (Maximum Strain failure theory)},\\
&= 10.94 \text{ (Tsai-Hill failure theory)},\\
&= 16.06 \text{ (Modified Tsai-Hill failure theory)},\\
&= 22.39 \text{ (Tsai-Wu failure theory)}
\end{aligned}
$$



- For a unidirectional lamina

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix} \tag{2.8.1}$$

Thermally induced strains:

$$\begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \tag{2.8.2}$$

Moisture induced strains:

$$\begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix} = \Delta C \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix} \tag{2.8.3}$$

- For a unidirectional lamina

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix} \qquad (2.8.4)$$

$$\Downarrow$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 - \varepsilon_1^T - \varepsilon_1^C \\ \varepsilon_2 - \varepsilon_2^T - \varepsilon_2^C \\ \gamma_{12} \end{bmatrix} \qquad (2.8.5)$$

2.8. Hygrothermal Stresses and Strains in a Lamina

- For an angular lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} + \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} \qquad (2.8.6)$$

Thermally induced strains:

$$\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix} \qquad (2.8.7)$$

Moisture induced strains:

$$\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} = \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix} \qquad (2.8.8)$$

- For an angular lamina

Transformation of CTE

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \qquad (2.8.9)$$

Transformation of Coefficients of Moisture Expansion

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \qquad [T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}$$

$$c = \mathrm{Cos}(\theta)$$

$$s = \mathrm{Sin}(\theta)$$

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Homework of chapter 2

2.1. Consider an orthotropic material with the stiffness matrix given by

$$[C]=\begin{bmatrix} -0.67308 & -1.8269 & -1.0577 & 0 & 0 & 0 \\ -1.8269 & -0.67308 & -1.4423 & 0 & 0 & 0 \\ -1.0577 & -1.4423 & 0.48077 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \end{bmatrix} \text{GPa}$$

Find:

1. The stresses in the principal directions of symmetry if the strains in the principal directions of symmetry at a point in the material are ε_1 = 1 μm/m, ε_2 = 3 μm/m, ε_3 = 2 μm/m; γ_{23} = 0, γ_{31} = 5 μm/m, γ_{12} = 6 μm/m
2. The compliance matrix [S]
3. The engineering constants E_1, E_2, E_3, ν_{12}, ν_{23}, ν_{31}, G_{12}, G_{23}, G_{31}

2.2. For a 60° angle lamina of boron/epoxy under stresses in global axes as σ_x = 4 MPa, σ_y = 2 MPa, and τ_{xy} = –3 MPa. The engineering elastic constants of the lamina are E_1 = 204 GPa, E_2 = 18.5 GPa, ν_{12} = 0.23, G_{12} = 5.59 GPa. Find the following

1. Global strains
2. Local stresses and strains
3. Principal normal stresses and principal normal strains
4. Maximum shear stress and maximum shear strain
5. The six engineering constants in global axes (E_x, E_y, m_x, m_y, ν_{xy}, G_{xy}).

2.3. Find the maximum biaxial stress, σ_x = –σ, σ_y = –σ, σ > 0, that one can apply to a 60° lamina of graphite/epoxy. The mechanical properties of a unidirectional graphite/epoxy lamina are found from Table 2.1 (Book of Autar K. Kaw, 2006). Use maximum strain and Tsai–Wu failure theories.

2.4. Find the off-axis shear strength and mode of failure of a 60° boron/epoxy lamina. Use the properties of a unidirectional boron/epoxy lamina from Table 2.1 (Book of Autar K. Kaw, 2006). Apply the maximum stress failure, maximum strain, Tsai–Hill, and Tsai–Wu failure theories.